# THE EFFECTS OF STATIC GRAPHIC, ANIMATED GRAPHIC, AND INTERACTIVE ANIMATED GRAPHIC PRESENTATIONS ON ACQUISITION OF THE TANGENT CONCEPT

By

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# THE EFFECTS OF STATIC GRAPHIC, ANIMATED GRAPHIC, AND INTERACTIVE ANIMATED GRAPHIC COMPUTER PRESENTATIONS ON ACQUISITION OF THE TANGENT CONCEPT

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The purpose of this study was to examine the relative effectiveness of static graphic, animated graphic, and interactive-animated graphic computer presentations of the tangent concept in imparting a mental image of the rotating secant and aiding understanding of the tangent concept. Also, this study was designed to examine any interactions between gender and presentation type. The image of a secant rotating to the limiting position of the tangent is typically part of textbook discussions of the tangent concept. Students viewed one of three different computer presentations, developed with Toolbook multimedia software, of the rotating secant image with accompanying text. The only difference in the three presentations was that one used a static graphic, one an animated graphic, and one an interactive-animated graphic (animation was controlled with the mouse) when presenting the image of the rotating secant. Since spatial visualization ability has been positively correlated with mathematics achievement, the Card Rotations test (Ekstrom et al., 1976) was administered to serve as a covariate. A test developed by the researcher to measure image and concept acquisition served as a posttest.

There was no statistically significant correlation between spatial visualization ability and success in acquiring the mental image of the rotating secant. Statistical analysis revealed neither treatment effect on acquisition of the rotating secant image nor any interaction between gender and treatment. Males outperformed females in measures of concept acquisition and image acquisition. The interactive-animated group outperformed the static group on the test for the tangent concept.

# CHAPTER 1
# DESCRIPTION OF THE STUDY

## Introduction

Visual imagery is used by both mathematicians and students to gain understanding of mathematical ideas (Dreyfus, 1991, Hadamard, 1954, Davis & Anderson, 1979). Textbooks attempt to convey standard visual images that may be helpful in this regard to the reader. Many of these images involve movement; the reader may be instructed to imagine motions, such as a point moving along a line or a line rotating about a point. Books, however, may not be the best medium for presenting some of these images because readers are not always able to form dynamic mental images that the author describes with text. Also, the reader often fails to attend to the pertinent aspects of the image (Orton, 1984). The animation and interactive capabilities of computers may provide a better means for conveying some images. This study compared the effectiveness of alternative presentations for one such dynamic image commonly presented in calculus textbooks--the image of a secant line rotating to the limiting position of the tangent line--in imparting a visual mental image and promoting a conceptual understanding of the tangent. Utilizing a personal computer, the three modes of presentation were static, animated, and interactive-animated. Three questions were addressed: 1.) Under what conditions does the student best acquire the image? 2.) Is the image helpful in understanding the tangent concept? 3.) Is there an interaction between gender and presentation type? Written tests were utilized to measure the constructs of image acquisition and schematization, and the student's understanding of the tangent concept.

## Statement of the Problem

Mundy (1984) posits that student understanding of calculus is superficial and restricted to mechanical aspects because students fail to "achieve a visual understanding of basic underlying notions" (p. 170). Monk (1988) also concluded that students do not have visual images for the main ideas in calculus. More specifically, Monk suggested that student understanding of graphs is static while an "across time" understanding of graphs is required in calculus. One approach to solving this problem is to place a greater emphasis on visual representations in the calculus courses (see textbooks by Hughes-Hallett & Gleason, 1997, Dick & Patton, 1994, Davis, Uhl & Portia, 1994, Ostebee & Zorn, 1998). Another avenue for enriching the students' visual understanding is to improve their spatial abilities

(Ferrini-Mundy, 1987) and thereby enhance their access to visual imagery.

This researcher is suggesting a third approach to improving students' visual understanding of calculus – using new technology to augment student visualization. Interactive animated computer presentations can act as metaphoric eyeglasses in helping students with poor spatial visualization abilities to see images that are important in learning calculus. This writer hypothesizes that the student lack of across-time understanding is partly a result of the inherent limitations of the static media, textbooks and blackboards, used to introduce temporal ideas. Perhaps concepts that require across-time understanding would be best presented using animated images. Textbook authors try to impart an across-time understanding by offering descriptions of moving images. Writers attempt this by displaying a graphic accompanied by text that asks the reader to imagine a moving point or line. Students' failure to develop across-time understanding may be the result of their inability to acquire mental images of these moving lines or points from static presentations.

But now educators have a new medium for presenting these images--personal computers, which can effectively display animated graphics that can be controlled by the user. The problem is whether success in acquiring dynamic visual images can be improved by viewing and controlling animated computer images. Also, is the student who successfully acquires the moving image associated with a concept more likely to achieve an understanding of the related concept? This study will investigate the effectiveness of the computer attributes in conveying moving images to the student. Relative comparisons will be made between the static presentations, animated presentations, and interactive-animated presentations.

## Visual Imagery and Spatial Ability

Lean and Clements (1981) called imagery "The occurrence of mental activity corresponding to the perception of an object, but when the object is not presented to the sense organ," and visual imagery is "imagery which occurs as a picture in the mind's eye" (p. 267-8). Dreyfus (1995) defined visual imagery as "mental images with a strong visual component ... that often derives from external visual information" (p. 3). There is disagreement over whether mental visual imagery takes the form of pictures in the mind or is more abstract. Shephard and Metzler (1971) provided strong evidence that the mental visual image is like a picture in the mind's eye (picture like). Chamber and Reisberg's (1985) research suggests that visual images are more abstract. The evidence so far indicates that some images are concrete and others are more abstract. Also, the capacity for vivid visual imagery varies among people (Galton, 1883).

In this study, “visual image” will be defined, as Norma Presmeg (1986a) suggested, as a “mental scheme depicting visual or spatial information” (p. 42). Images involving movement and the ability of the participants to imagine motion in static images will be one focus of this investigation. The perceived object may be present, but the motion will not be supplied when testing for acquisition of dynamic mental images. Visual imagery, in this case, is consistent with Presmeg’s definition.

Mathematicians utilize visual imagery in their work (Hadamard, 1945; Papert, 1980; Davis & Anderson, 1979). Students also make use of visual imagery when constructing meaning about mathematical concepts (Duval, 1995; Moreno & Sacristán, 1995). Bishop (1989) argued that “visual presentations offer a powerful introduction to the complex abstractions of mathematics.” Tommy Dreyfus (1991 maintained that “visualization is generally considered helpful in supporting intuition and concept formation in mathematics learning” (p. 33). Battista (1982) found that cognitive development is more strongly correlated with success in geometry than is spatial ability, but both attributes are important in understanding geometry. MacFarlane Smith (1964) claimed that visual imagery is essential to learning mathematics. It follows that one of the goals of instruction is to help students acquire visual images that foster an understanding of mathematical concepts. The main approach, in this regard, has been to find ways to improve students' visualization abilities, which Lohman (1988) defined as "the ability to generate, retain, and manipulate abstract visual images" (p. 188).

MacFarlane Smith (1964) suggested that measures of spatial ability must "involve the perception, retention and recognition (or reproduction) of a figure or pattern in its correct proportions" (p. 96). McGee (1979) hypothesized two types of spatial ability: spatial orientation and spatial visualization. The former is involved in understanding images where the perspective of viewer is changed, but the image remains static. The latter is involved in tasks that require mentally rotating or moving visual images. Tartre (1990) suggested that the Card Rotations test (Ekstrom et al., 1976), the Form Boards test (Ekstrom et al., 1976) and the Space Relations section of the Differential Aptitude Test (Bennett, Seashore, & Wesman, 1973) are measures of spatial visualization ability. Subjects in the present study were required to mentally rotate secant lines and predict changes that result from these rotations on a static diagram. Because spatial visualization is required to perform these tasks, the Card Rotations test was administered early in this study to determine its usefulness as a covariate.

The results on improving spatial ability have been mixed (Baker, 1990). This researcher proposes another avenue for improving student use of visual imagery in mathematics--making images more accessible to students through the use of new technology. This approach is particularly promising for dynamic images. Dynamic images in mathematics books are described textually; this requires that the student must translate a linguistic rendering into a visual image. It is plausible that some students have trouble with this translation and may benefit from viewing animated images.

Mathematics textbooks present diagrams accompanied by instruction for interpreting the diagram in order to impart visual images to the reader. These are images that educators have presumed help students understand concepts; the images have become institutionalized by their use in textbooks and by teachers in the classroom. For example, the graph in Figure 1 can be useful in helping a student to understand the Mean Value Theorem:

> If a function $f$ is differentiable on $(a,b)$ and continuous on $[a,b]$, then there is a number $c$, $a<c<b$, such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

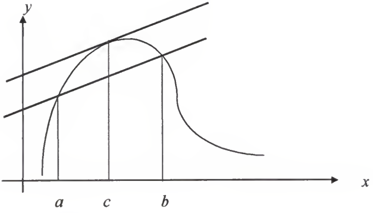


Figure 1. Graph of $y = f(x)$ illustrating the Mean Value Theorem.

The graph is helpful because the reader is provided with the spatial relationship between secant line segment $\overline{ab}$ and the tangent at point $(c, f(c))$--they have the same slope (apparent by a visual inspection). The theorem states that there is a point on the graph between $x = a$ and $x = b$ where the slope of the graph equals the slope of the secant line.

## Action Metaphors

Textbooks attempt to instill visual images with graphics and textual descriptions. However, textbooks are limited to presenting only static graphics. Although static graphics have been shown to enhance understanding (Dwyer, 1972, 1978), animated graphics may provide better representations of some mathematical concepts, particularly those that have action metaphors. The term action metaphor will be used to describe dynamic images that are useful in imparting understanding to static mathematical concepts. Action metaphors, like metaphors in general, make understanding in one domain (in this case, the static and abstract) more accessible through understanding in another domain (the dynamic and concrete). For example, when justifying the formula for the area of a parallelogram, the teacher may ask the student to imagine cutting the figure appropriately and moving one of the triangles to the other end of the parallelogram to make a rectangle. Although the formal derivation of the area formula involves no moving triangles, this explanation describes movement. The calculus instructor, while discussing the area function, uses an action metaphor when asking the student what happens to the area between the curve and an interval on the $x$-axis, as the right endpoint of the interval moves along the $x$-axis. While it is true that the area is a function of x, the definition of this function does not contain any reference to movement. Dynamic graphics used in application of mathematics are typically not action metaphors – they are usually direct representations. For example, the description of falling bodies often used when discussing the derivative would not be considered an action metaphor – it is a mathematics application for a dynamic process.

## Examples of Action Metaphors

E. Richard Heineman (1988), in his textbook Plane Trigonometry, explained variations of the sine and cosine curves by describing the change in the length of one leg of an inscribed right triangle as the hypotenuse sweeps around a unit circle. He wrote, "As $\theta$ varies, *make P move* so that r is always equal to 1. This will keep P on a circle with center O and radius 1. . . .Now try to *visualize $\theta$ as increasing* from $0^{o}$ to $90^{o}$ (p. 98)" (italics added).

When discussing increasing and decreasing function in College Algebra, Larson, Hostetler, and Munn (1992) wrote, "As we move from left to right, this graph falls from $x=-2$ to $x=0$ . . . " (p. 271). The authors also use action metaphors when they refer to "folding the Cartesian plane (p. 201)", "shifting" graphs (p. 281), and "stretching" graphs (p. 284).

In Calculus With Analytic Geometry, Howard Anton (1992) described a tangent to a graph: "If a secant line is drawn between points P and Q , and Q is allowed to move along the curve toward P, then we can expect the secant line to rotate toward a 'limiting position' . . ." (p. 162).

This purpose of this study is to investigate the effectiveness of animated graphics and interactive-animated graphics in imparting a mental image when presenting an action metaphor in calculus. Monk observed that students lack a visual image of the basic notions in calculus because they fail to develop an across-time understanding of graphs. Ferrini-Mundy (1991) has found that the students' concept of function is static because they think of functions one point at a time. This writer contends that dynamic imagery is required to overcome these shortcomings and, for many students, development of dynamic images can be enhanced by experience with animated computer representations of action metaphors. The limitations of the static mediums of our primary means of communicating images, textbooks and chalkboards, may contribute to the static image that many students have of functions. Since students never see dynamic representations of functions, and are often unable to conjure up the dynamic images described in textbooks, they may be inclined to view functions in a static, pointwise manner.

## The Tangent Concept

The concept definition of tangent is static, but its conception is dynamic. Newton rejected a static notion of lines when discussing tangents, "I consider mathematical quantities in this place not as consisting of very small parts; but as described by continuous motion" (Struik 1969, p. 303). He described lines as objects generated by moving points, surfaces as generated by moving lines, and solids as generated by moving planes. This interpretation of geometrical objects as generated by moving objects is intuitive and derives from perceptual experience. Instantaneous rate of change can be defined analytically as the limit of the difference quotient, or intuitively as the visual-spatial slant or inclination of a tangent line. Modern textbooks use dynamic imagery to help the student make a connection between the intuitive notion of tangent and the analytical definition of tangent. The visual representation provides the student with a kind of gestalt that makes the analytical representation comprehensible.

Vinner (1991) makes a distinction between concept definition and concept image. The concept definition is formal and accepted by the mathematical community as complete. The concept definition of "tangent line" is the line through the point $(x_1, y_1)$ having slope:

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}.$$

The concept image is a non-verbal representation of the concept that consists of all the students' "visual representations, the mental pictures, the impressions and the experiences associated with the concept name" (Vinner, 1991; p. 68). The rotating secant, the tangent, slope of the tangent, slope of the secant, $\Delta x$, and $\Delta y$--all have both visual and analytical representations. The slope of the secant approaches the slope of the tangent as $\Delta x$ approaches zero. This can be made visually explicit in computer animation by using a thickened line to represent $\Delta x$ and a rotating line to represent various secant lines associated with $\Delta$x. I am making the distinction between spatial relations and analytical relations. Students apprehend many mathematical concepts by alternating between the visual and analytical representations. This will be further discussed in Chapter 2.

The complete image of the rotating secant, including the spatial relationships between the moving parts of a dynamic image, should ultimately help the student understand that the

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{slope of tangent line}$$

(both visually and analytically). This notion will eventually be generalized to the derivative. The problem is that students are not acquiring the image (Orton, 1977, 1983).

Another image that is used to present the tangent concept is that of zooming in on a curve. Since differentiable curves look locally straight when sufficiently magnified, the slope of a curve at a point is approximated by the slope of the apparently straight line upon magnification of the curve. Graphing calculators or graphical software is used to help the student develop the image of zooming in on a graph. This writer believes that the rotating tangent image is a better precursor to the notion of instantaneous rate of change and its generalization to the derivative. The image of zooming in on a graph to see its local linearity leaves implicit the notion of $\Delta$x approaching zero.

There is a cognitive jump in seeing that as one zooms in on a graph, the scale changes; hence the change in x in the magnified view has decreased. In the rotating secant image, the change in x is explicit. Another problem with zooming in on graphs is that it can be disorienting (Goldenberg, 1988). Some students, for example, think that they are viewing a different graph after zooming in. The image of local linearity may have become a common approach to inspiring the tangent concept and slope of a curve because, first, graphing calculators made the image more accessible, and second, the image of the rotating tangent line is not easily conveyed to the student using static media. A comparison of the two approaches in helping students apprehend the tangent concept is suggested for further study.

Figure 2 illustrates a typical graphic presentation of the tangent concept via a rotating secant. The student is to infer that as point Q approaches point P, h (see Figure 3), an incremental change in the abscissa, diminishes and the secant becomes a closer approximation of the tangent. It is said that the secant line rotates toward the tangent as h approaches zero.

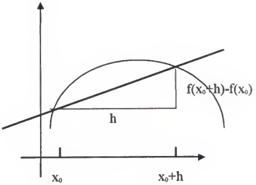


Figure 3. Secant line.

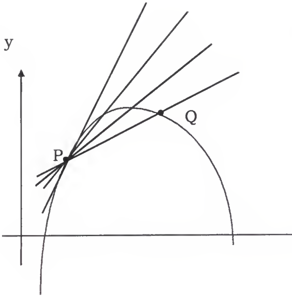


Figure 2. As point Q approaches point P, the secant line approaches the tangent line.

Research has shown that students rarely see the picture this way (Orton, 1977, 1983). I contend that an animated graphic of this limiting process will help the student develop a more accurate mental image of the secant tending toward the tangent as h approaches zero.

This study compared several graphic approaches to improving student visualization of the tangent concept. The tangent to the graph $y = f(x)$ at point $(x_1, y_1)$ is the line through the point $(x_1, y_1)$ having slope $m = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$. (In many textbooks, Δx is used in place of h.) A clear understanding of this is necessary for making sense of almost all calculus concepts, including the derivative and the Fundamental Theorem of Calculus. Walter Zimmerman wrote:

> One of the major difficulties with the calculus curriculum is that it has become a landscape without landmarks. . . . Many [students] are wandering through a forest examining one tree, then another and another. All trees seem pretty much the same. None stands out above the others. There are no clear paths through. There are no landmarks. They never reach a clearing where they can see the forest as well the trees. . . . A repertoire of visual images, carefully selected and well understood, may provide landmarks of meaning and progress. Such images can provide structure to the subject as a whole and integrate various components of a particular topic. One could argue that visualization and visual thinking should be one of the central elements in calculus reform. (Zimmerman, 1991,p. 135-6)

While Zimmerman is expressing a sentiment that is oft expressed by calculus educators, it is interesting that he used a visual metaphor when describing students' lack of understanding in calculus – students become embroiled in the minutia and miss the big picture. Vision is a common metaphor for understanding. This is evidenced by such common sayings such as:

*I see what you mean.*

*She has good insight into the matter.*

*Don't muddy the issue.*

*Let me show you what I mean.*

*I get the picture.*

*What is your point of view on this issue?*

*She has a different perspective on this matter.*

Indeed, Isaac Newton used a visual metaphor when he said, "If I have seen further than others, it is because I have stood on the shoulders of giants."

A clear understanding of the tangent concept should be a landmark for every calculus student. The student needs to develop a concept image that includes the analytical description of a generic tangent as well as visions of the secant line rotating to the limiting position of tangent as h approaches zero. The secant is described as a function of h. That is, h is the independent variable and the resulting secant line is the dependent variable. A dynamic image of this function will aid understanding the tangent concept. Multimedia software and more powerful computers have made dynamic presentations of the tangent concept and other mathematical concepts possible. In this investigation, three groups studied the tangent concept. One group saw a static presentation; another viewed an animated presentation, and the third viewed an interactive animated presentation.

## Hypotheses

1. The animated group will out-perform the static group in image acquisition. The image will be said to be "acquired" if it is

remembered and generalized to smooth curves in different positions.

2. The interactive-animated group will out-perform the animated group in image acquisition.
3. Females will benefit more than males from animated presentations in image acquisition.
4. Females will benefit more than males from the interactive animated presentation in image acquisition.
5. Those that acquire the image will develop the most complete concept image of the tangent, which will be measured by the student's ability to draw tangents to curves and determine where tangents do not exist.

## Bases for Hypotheses

### Hypothesis #1

Animated presentations should be more effective in imparting a dynamic image because they will provide a more concrete representations of a secant approaching a tangent as the change in the independent variable approaches zero, thus helping the student to develop a powerful and vivid mental image. The student will not be required, as in static presentations, to translate a linguistic description into a visual image. While the student may not imagine the movement described using text and a static graphic, an animated demonstration will allow the student to see the movements of the secant line and Δx.

Hypothesis #2

The interactive-animated group will control the animation by dragging the mouse along the x-axis to change Δx. This group should out-perform the animated group, who has no control over the animation, for several reasons.

1. More Control:

The students can experiment by increasing and decreasing Δx at will. Participants might first concentrate on the relation between Δx and Δy, then repeat the movements to concentrate on the relation between Δx and the secant. Finally, they can attempt to make sense of the relationship of all four variables (Δx, Δy, secant line, and point Q) simultaneously. They can repeat the animation, all or part, as many times as they wish. They can run it forward and backward, faster and slower. The students will benefit from the ability to adjust the speed of the animation because the rate at which the variables change can be a clue to their relationships. Move the mouse faster and the secant line rotates faster – speed provides another mode for communicating a relationship between Δx and the secant position.

2. Directed Attention:

Understanding the definition of "tangent" requires the student to think about the difference quotient as Δx approaches zero--an analytical process. Understanding the tangent concept also requires that the student think about what happens to the secant line and its slope as $\Delta x$ approaches zero--a visual process. The slope of the secant line is a function of $\Delta x$; in the interactive presentation, the students' attention is directed to the independent variable, $\Delta x$, because the student controls the animation by changing $\Delta x$.

3. Visual attention is freed up:

It is difficult to visually keep track of four moving objects simultaneously. But if the students can employ another sense, tactile, to monitor the movement of one of the objects, the other three will be easier to follow. The participants can keep track of the size of $\Delta x$ without looking at it when $\Delta x$ is controlled by the mouse. For example, they will know $\Delta x$ is increasing when they are moving the mouse to the right. Consequently, each student's gaze can be directed to $\Delta y$, point Q, and the secant line.

4. Kinesthetic memory:

The goal of presenting the rotating secant image is not only that the students can understand the relationships of the parts of the image, but also that they can recall the image when thinking about related concepts, such as the derivative or the slope of a curve. The student will have a kinesthetic memory as well as a visual memory of the event. What Howard Gardner (1983) called "bodily-kinesthetic" and

Jerome Brunner (1967) called "enactive" are ways of knowing that involve the human musculature.

5. More concrete:

The image will be more concrete and more likely to resonate with bodily experience. Part of our everyday experience involves pushing or pulling, which results in an action. For example, when we push or pull on a doorknob, the door rotates. When the student pushes or pulls $\Delta x$ (with the mouse), point P is pushed or pulled along the curve. This action, which results in the rotation of the secant line as $\Delta y$ changes, associates the daily experience of motions with the abstract notions of mathematics.

6. Better engagement:

When the student is actively involved in the animation he or she gets drawn into the figure. As will be discussed in Chapter Two, the passive observer of computer animation often attends to the wrong aspects of the visual. When the student is actively involved in the animation, he or she will be more attentive and better able to understand the relationship of the motions.

7. Integration of senses:

"The integration of two or more senses contributes to a more robust and memorable experience ... maximizes the potential for significance, meaning, and re-membering (sic). (Mason, J., & Heal, B., 1995, p.

293)" Howard Gardner (1983) agrees that using two or more types of intelligence improves ones understanding of new ideas.

Hypotheses #3,4

Males outperform females on tests designed to measure spatial ability (Maccoby & Jacklin, 1974; Harris, 1981; McGee, 1979). Spatial ability is involved in forming visual images (Lohman, 1979). I posit that there will be an interaction between gender and treatment. Smith (1972), in a review of the literature on spatial ability, suggests that spatial ability is positively related to visual imagery. If high spatial ability is required to imagine the movements described in static diagrams, then females will benefit more than males from the more explicit animated presentations. The computer animation will help females overcome lower spatial ability by making the image more accessible. The student must first see the image before remembering and schematizing it. The computer can act as a crutch to one's spatial ability by making some dynamic images more concrete and thus more attainable. A study by Acker and Klein (1986) supports this notion. They found no gender difference in the spatial tasks of estimating where the end of a line segment would end up if rotated ninety degrees, and estimating the combined length of two sides of an isosceles triangle when participants were allowed to view an animated presentation of a partial rotation of the line or a partial straightening of the two sides of the triangle.

Spatial ability is positively correlated with mathematics ability (Conner and Serbin, 1985; Smith (1964); Guay & McDaniels (1977); Sherman, (1979). I posit that this is partly because high spatial ability allows one to acquire visual mental images readily. When the student thinks productively about a mathematical concept, he or she alternates between visual and analytic thought (Zaskis et al., 1996). The student who fails to "see" an image is deprived of the visual side of mathematical thinking and is forced into memorizing the analytic according to a system of meaningless rituals, and the result is the student over-relies on the analytical.

### Hypothesis #5

In order to understand the definition of "tangent," one must coordinate the changes of $\Delta x$ approaching zero while $\Delta y$ decreases and the secant line rotates to a limiting position. As $\Delta x$ approaches zero, $\Delta y/\Delta x$ approaches a limit (a number) that corresponds to the slant (an image) of a line. The dynamic image of a rotating secant will result in a gestalt that will allow the student to synthesize the separate components ($\Delta x$, $\Delta y$, slope, secant) into an integrated, structured whole instead of seeing these as separate, unrelated elements.

Once having acquired the image, the student can call it up at will to examine it. The student can answer “what if” questions, such as “What if point P is located at a discontinuity or a corner?” All of this analysis will be carried out according to the Visualization/Analytic Model (Zaskis et al., 1996), which requires student access to the visual aspects of the concept (discussed in Chapter Two). For these reasons, I hypothesize that those who most successfully acquire the image will best understand the concept of tangent.

## Significance of This Study

This study concerns the use of computer technology in visualization of dynamic images when learning calculus concepts. This impacts education and educational research in six important ways.

1. Most educational research in visualization has focused on visualization in problem solving. This study extends the body of research into visualization in learning abstract mathematical concepts.
2. Many educational studies in visualization has focused on improving students’ spatial ability. This study extends the body of research into the effectiveness of tools that supplement students’ spatial ability (as a microscope supplements the visual ability).

3. Most studies in visualization of mathematical concepts have involved the use of static images. This study focuses on the acquisition and use of dynamic images.
4. Student reluctance to visualize in calculus is well documented. This study will add to the body of knowledge about student visualization in calculus.
5. The concept under investigation, tangent, is basic to one's understanding of derivative, and hence, to all differential and integral calculus. An improved method of teaching this concept will improve calculus learning.
6. This study has the potential to find ways in which the use of computers in mathematics education can be improved through the applications of dynamic and interactive capabilities. Such new information has the potential not only for directing the most effective curricular use of computers, but also for contributing to the interdependent processes of software development, hardware development, and educational applications of the computer.

# CHAPTER 2

# LITERATURE REVIEW

## Theoretical Background

Both analytical and visual reasoning are required for understanding most mathematical concepts. This is particularly true for apprehending calculus concepts. Descartes' marriage of geometry (a highly visual subject) and algebra (a predominantly analytical subject) made calculus, as we know it today, possible. Regarding visual considerations, Shlomo Vinner (1989) stated, "They are indispensable in the calculus course" (p. 55). As for analytic reasoning, Krutetskii (1976) argued that analysis and logic are required for all mathematical thinking. The following section will introduce literature that supports the proposition that both modes of thinking are necessary and that an alternation between visual and analytic thinking is present in most successful mathematics learning.

Visual thinking is associated with intuition (Miller, 1996; Tartre, 1990), while analytical thinking is identified with rigor, formal mathematics and logic (Krutetski, 1976; Davis, 1979). Some argue that mathematical thinking is primarily non-analytical and that the analytic is used to prove and communicate the results (Davis & Anderson, 1979; Hadamard, 1945). Zaskis, Dubinsky, and Dautermann (1996), who take issue with this dichotomy, propose the Visualization/Analytic (VA) model, in which visual and analytic methods are used by the learner in concert to construct meaning of mathematical ideas. Thinking is described as an iterative process that begins with visualization ($V_1$) (see Figure 4) followed by analysis ($A_1$). After the first analysis, the subject returns to the same image ($V_2$), but the picture takes on new meaning as a result of the first analysis: "... each act of analysis, based on the result of the previous act of visualization, is used to produce new, richer visualization, which is then subjected to more sophisticated analysis" (p. 447). The alternation between visualization and analysis enhances meaning with each iteration.

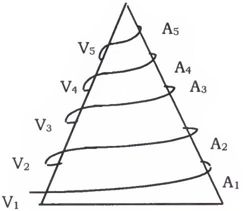


Figure 4. Visualization/Analysis Model (Zaskis et al., 1996)

Initially, the movement between the visual and the analytic is difficult for the student. After several iterations, the visual and analytic thinking become more closely integrated and the two modes become almost indistinguishable. This notion of a narrowing of differences between the visual and the analytic is consistent with Presmeg's (1992) hypothesis of a concrete to abstract continuum of visual imagery. At the extreme end of the scale is pattern imagery that is highly schematic and may not be recognized as imagery by an observer. Initially, visualization in the VA model is concrete, and subsequent iterations lead toward a more abstract pattern imagery.

Zazkis et al. support the VA theory by analyzing interviews with students after taking an introductory abstract algebra test. The students' task was to identify the elements of the dihedral group $D_4$ and find the products of various elements in the group. Two methods for this were suggested during the course: a visual approach, involving transformations (rotations and flips) of a square with numbered vertices, and an analytical approach using multiplication of permutations. After taking a written test, 32 students described their methods of solutions to researchers in individual interviews. Most students used a combination of both methods. In a typical act of initial visualization, the student rotated a square, followed by the analytical act of naming the vertices. The student then performed a more sophisticated visualization by rotating the square and observing the motion of the vertices, followed by an analytic act of mapping numbers to numbers – for example $(1,2,3,4)\xrightarrow{R_{180}}(4,3,2,1)$. The next visualization consisted of performing two operations on the square and attempting to find a single, equivalent operation. Each act of visualization became more sophisticated with the help of the previous act of analysis; and each act of analysis became more refined with the help of the previous act of visualization.

Another researcher who observed an interplay between the visual and analytic in students' thinking showed subjects, ages 10-18, either a cube, a parallelepiped, a triangular prism, or a regular tetrahedron (Moriotti, 1995). After the solid was removed from sight, the participants were asked to tally the number of faces, vertices, and edges. The researchers analyzed the students' approaches to these mental exercises and found that many students were able to enumerate these features by counting directly from a visual mental image of the object. However, most students found this approach difficult because they lost track in the counting process, and approached the problem using both visual images and concepts about the object. Mariotti described it as a "dialectic process between two aspects, the figural and the conceptual" (p. 108). Maroitte's "figural aspect" is consistent with the Zaskus' et al. "visual," and her "conceptual aspect" is consistent with analytic side of the VA triangle. For example, one subject counted the edges of cube by first visualizing "the end of three faces, that join in one tip," followed by an act of analysis "there are three of them (edges) coming out of a tip, then $3\times 4=12$" (p. 108). Mariotte's description of thinking is consonant with the Visual/Analytic model but, unlike the Zaskis et al. study, the visual aspect is imaginal – the subjects were required to recall images from previous perceptual experience.

## Concept Image, Concept Definition, and the VA Theory

Mathematics is a deductive science where necessary conclusions are reached from a set of axioms and definitions. However, mathematics is generally created by a process of induction (Davis & Anderson, 1979). The creative mathematician may propose generalizations through an inductive process; then use deduction only in the end to prove the results. Likewise, students do not learn mathematics in a purely deductive manner of definition-axiom-theorem-proof. The first obstacle in this chain of reasoning is that of definition. Students are rarely able to fully understand a complex mathematical definition at first reading. Tall and Vinner (1981) introduced the notion of concept image in order to shed further light on the process of understanding mathematical definitions. While the concept definition is the formal definition of a mathematical concept, the concept image includes all the mental images and ideas about a concept that are built up over years of experience by the student. For example, the concept definition of "tangent" to a curve at point P is the line through P with slope equal to the limit of the difference quotient as $\Delta x$ approaches zero. The concept image, which is evoked by the concept name tangent, is a nonverbal collection of all the images and experiences associated with that concept. The concept image is unique for each individual. The concept image for tangent may include images of:

Tangents to a circle
Tangents to a parabola
Rotating secant lines
Zooming in on curves
Falling objects (instantaneous velocity)
Difference quotients
Limits
Functions
Point-slope form of lines
Slope
Tangents at inflection points
Tangents coincident with graphs
Vertical tangents
Tangents at relative extrema
Derivatives

Elements of the student's concept image will often conflict with the concept definition. For example, a student's concept image of tangent often includes the image of lines that touch, but do not cross, curves. This image is in conflict with the concept definition of tangent that includes tangents that cross curves. This misconception may cause no immediate problem for the student in his or her study of calculus. For instance, the tangents at extrema (a central topic in calculus) do not cross the curve (see Figure 5). In this case, the misconception is called a “potential conflict factor” (Tall & Vinner, 1981). At relative extrema, the student may encounter the conflict where a tangent does indeed cross the curve (see Figure 6). The potential conflict factor is then called a “cognitive conflict factor” (Tall & Vinner, 1981), which may cause a sense of unease in the student. He or she may attempt to resolve this conflict by referring to the concept definition, or by drawing a shortened tangent line (see Figure 7). If the student refers to the concept definition in the above conflict, he or she has a chance in improving his or her concept image of tangent. Students who draw a shortened tangent line will achieve a resolution that may be sufficient until the student encounters a new, unresolvable conflict such as a tangent at an inflection point (see Figure 8). The success with which a student acquires a concept can be equated with the degree with which his or her concept image agrees with the concept definition.

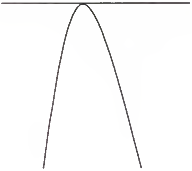


Figure 5. Tangents at extrema do not cross the curve.

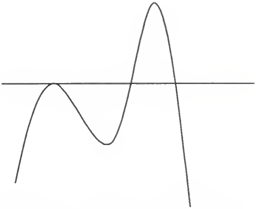


Figure 6. Tangents at relative extrema that cross the curve can result in a cognitive conflict factor.

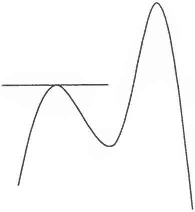


Figure 7. Student may resolve cognitive conflict factor by drawing a shortened tangent line.

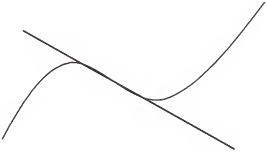


Figure 8. Tangent at an inflection point may result in an unresolved cognitive conflict factor.

Davis and Vinner (1986) theorize that a successful approach to solving a problem involving mathematical concepts involves an interplay between concept definition and concept image as illustrated in Figure 9.

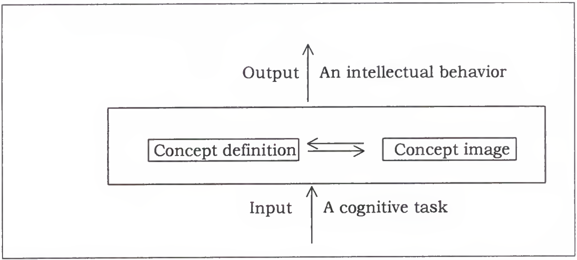


Figure 9. Interplay between concept definition and concept image. (Vinner, 1991, p. 71)

The theory of concept image is consistent with the VA Theory. First, the theory of concept image acknowledges the importance of the visual aspects of the concept image and the analytic aspects of the concept definition. Second, the theory posits that one's concept image begins with an incomplete understanding including conflicts and misconceptions. Upon further reflection and experiences, the concept image can progress toward a more complete and accurate concept image that is consonant with the concept definition. The VA theory suggests that the resolution of conflict between conflict image and concept definition is brought about by alternation of thinking about the visual aspects (part of concept image) and analytic aspects (the concept definition) of the concept. Vinner (1991) suggested that as this alternation continues, the concept image and concept definition become more closely aligned; Zaskis et al. assert that as the alternation continues the visual and the analytic merge.

## Visual-Analytic Preferences

Most of the studies in mental imagery during mathematics learning have involved either student preference for imagery in problem solving or improving spatial visualization abilities. The following two sections will outline relevant research in the areas of student preferences and spatial training.

Moses (1977) measured a student's "degree of visuality" by the use prevalence of graphs, tables, diagrams and pictures in written solutions to problems. Suwarsono (1982) scored students on an "analyticality-visuality" scale by asking them, after they worked on a set of problems, to select from prepared sets of solutions the approach that was most like the one they used. Clements (in Lean and Clements, 1981) classified students as "analytic," "visual," or "harmonic" thinkers according to the amount of imagery they used to explain their solutions to problems during personal interviews. In a 12-year study, Krutetski (1976) categorized students as "geometric type," "analytic type," or "harmonic type" based on factorial analysis of test scores as well as observations of students during classes and problem solving, and interviews with them and with their parents, teachers, and friends.

Using an instrument she developed to measure student preference for visual methods in solving mathematical problems, Norma Presmeg (1986a), studied fifty-four senior high school students identified as visualizers: "individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and non-visual methods" (Presmeg, 1985, pp. 90-91). Presmeg classified five types of visual imagery and counted the number of students who use each type of imagery when solving a set of problems (Table 1).

Table 1. Five Types of Visual Imagery Used by Presmeg's Students.

| Type of imagery | Number of Students |
|---|---|
| concrete pictorial imagery (pictures-in-the-mind) | 52 |
| pattern imagery (pure relationships depicted in a visual-spatial scheme) | 18 |
| memory imagery of formula | 32 |
| kinesthetic imagery (imagery involving muscular activity) | 16 |
| dynamic (moving) imagery | 2 |

Presmeg also rated the teachers according to their propensity to use visual imagery. She found that the students who were visualizers benefited most from instruction by teachers who were in the middle group. These findings indicate that an over-reliance on visual methods can be detrimental to mathematical thinking. Visualizers did not perform well in classrooms with low visualizer teachers or with high-visualizer teachers because perhaps the instruction was overly analytic in the former class and overly visual in the latter. Another finding was that the visualizers in all classes had difficulty using mathematical language, remembering terminology, and communicating their ideas. This is evidence that they were weak in analytic skills. These students likely preferred visual modes because they were weak in analysis. The VA theory posits that both modes of thought are necessary. The problem for Presmeg's visualizers appears to be an over-emphasis by the students on the visual instead of a healthy synthesis of the visual and the analytic aspects of mathematics.

In another study by Presmeg (1986b), 13 teachers identified their best math students. Presmeg found that, according to her scale of mathematical visuality, none of the selected students were visualizers. On the surface, Presmeg's two studies might indicate that a visual approach to mathematics inhibits learning. But there are several reasons why the gifted students may have registered as nonvisualizers: First, high school mathematics teaching emphasizes and rewards analytical approaches (Moses, 1977; Dreyfus, 1991; Presmeg, 1985), so the students who were identified as the "best" were those who approached mathematics analytically. Vinner (1989) suggested that the preference for an analytic approach is learned and that it develops mainly because school mathematics overemphasizes the analytic approach. Second, the students may have found the problems trivial and may not have required visual methods. Students are incline to use analytic processing when solving familiar problems because the compact nature of analytic representations allows students to retain more information in short-term memory; (Johnson-Laird, 1972). Presmeg's (1986b) finding that visual methods take more time than non-visual methods supports this. Third, people are often unaware of their own visualization (Richardson, 1969), so the subjects could not report a visual solution to the researchers.

Lean and Clements (1981) also found that students who preferred analytic approaches to problem solving outperformed the more visual types. But they also reported that the analytic types scored higher than the visual students did on spatial tests. A study by Fennema and Sherman (1977) revealed a positive correlation between spatial ability and mathematics achievement.

No studies have compared calculus students with different preferences for visual reasoning, but researchers have found a general tendency for these students to avoid visual considerations (Vinner, 1989). This occurs for several reasons: interpreting images is difficult (Dreyfus & Eisenberg, 1990), students are taught that visual approaches are not valid, calculus curriculums under-emphasize visual approaches (Hughes Hallett, 1991), and students learn to pass tests by memorizing algorithms and analytic proofs (Vinner, 1989).

## Spatial Training

Because visualization is important in mathematical learning, some researchers have investigated the possibility of improving spatial ability through training. Of particular interest is a study by Joan Ferrini-Mundy (1987), who designed two 8-week, slide-tape spatial training programs for calculus students. These programs addressed two- and three-dimensional spatial tasks and area estimation. One program introduced spatial strategies through the use of visual representations on slides; the other program supplemented the slide program with manipulatives. The researcher compared two treatment groups, audiovisual and audiovisual-tactile, and a control group that received no spatial training. She found no significant differences among the groups in calculus achievement or improvement in spatial ability. She did find, however, two interactions of gender by treatment: 1.) Combining the two treatment groups and using measures of precalculus background and calculus background as covariates, the women in the treatment groups outperformed the women in the control group for a measure calculus achievement--the opposite was the case for men. 2.) The women in the audiovisual group outperformed the women in the audiovisual-tactile group in a measure of visualization of solids of rotation--the opposite was the case for men. Ferrini-Mundy concluded that, for some topics in calculus, spatial training might improve calculus learning more for women than men.

If Ferrini-Mundy is correct about the differential effects of spatial training for women and men, then it is likely that spatial support such as animated presentation of the secant line tending toward the tangent will benefit women more than men. The women in Ferrini-Mundy's study received spatial training, in part to improve their ability to mentally manipulate visual images. Because the student is required to mentally rotate the secant line in static presentations, animated presentations will be more effective for those who lack this skill.

## Prototype Phenomenon

Although mental imagery is generally considered helpful in learning mathematics, imagery can also introduce some obstacles to learning. Students often fixate on a prototypical image when thinking about a mathematical concept or definition. For example, when a definition of a right triangle is presented with a drawing of a triangle with one leg plumb with the page (the standard textbook orientation), students will often seize on that image and have difficulty identifying right triangles in non-standard positions. Hershkowitz (1987) found a strong prototype phenomenon when students failed to identify right or isosceles triangles in different orientations than the prototype. Providing half the students in each group with verbal definitions of each concept, along with visual examples, had no effect on success.

In another study, Hershkowitz (1989) found that the prototype phenomenon is prevalent even when no examples of a definition are provided. She made up two definitions of geometrical figures: "Bitrian"--a shape consisting of two triangles having a common vertex, and "biquad"--a shape consisting of two quadrilaterals having a common side. When college students were asked to identify or draw an example, they typically drew or identified the most popular types of example. Hershkowitz identified these popular examples as instances of the prototype phenomenon in which the subjects considered "one fixed shape was **the** concept" (p. 67); an incomplete concept image resulted. The key finding here is that the prototype effect occurred even though no visual examples were given with the definitions; the presentations of the definitions were entirely propostional. Further, she proposed that this is evidence that "visual-perceptual elements also play some role in the concept formation" (p. 67).

The conclusion of both of these studies is that although visual thinking is a necessary part of geometrical thinking, visual images can hinder geometrical thought as well because the prototype effect often limits the concept image.

There is evidence that the prototype phenomenon is present in learning the tangent concept. The tangent to a circle is usually introduced in a geometry course with a figure such as the one in Figure 10.

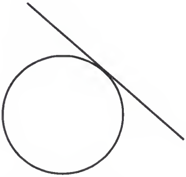


Figure 10. The prototypical tangent.

Tangents to a circle touch the curve but do not cross it. In the more general definition of a tangent presented in the calculus course, tangents cross the curve at points of inflection, are coincident with straight lines, and do not exist at points of nondifferentiablity. Vinner's (1982) study, in the next section, provides evidence that many calculus students have a limited notion of the tangent concept consistent with the prototypical image of a tangent to a circle.

## Student Understanding of Tangent Concept

The prototype phenomenon is an obstacle to student learning about the tangent concept. In a study done by Vinner (1982), calculus students were asked to draw tangent(s), if possible, at point P to the curves in Figure 11.

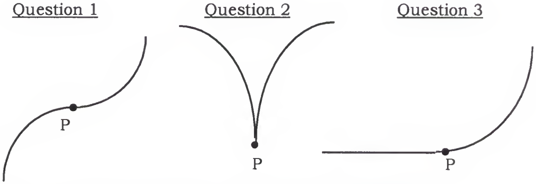
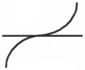
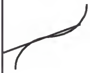
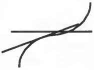


Figure 11. Students were asked to draw tangents at point P.

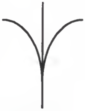
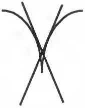
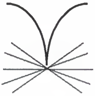
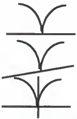
The results in Table 2, Table 3 and Table 4 indicated a large proportion of responses that could be explained by student fixation on the image of tangent to a circle.

Table 2. Distribution of Student Drawings to Question 1 (N=278).

| A | B | C | D | E |
|---|---|---|---|---|
| <br>The right answer | <br>A generic tangent | <br>Two tangents | Another drawing | No drawing |
| 18% | 38% | 6% | 10% | 28% |

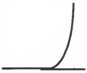
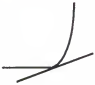
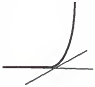
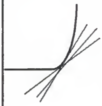
From Vinner, 1991.

Table 3. Distribution of Student Drawings to Question 2 (N=278).

| A | B | C | D | E |
|---|---|---|---|---|
| <br>The right answer | <br>two tangents | <br>infinitely many tangents | <br>A 'balance' tangent | No drawing |
| 8% | 18% | 18% | 14% | 42% |

From Vinner, 1991.

Table 4. Distribution of Student Drawings to Question 3 (N=278).

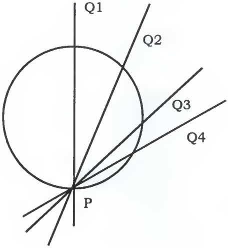
| A | B | C | D | E | F |
|---|---|---|---|---|---|
|  The right Answer |  A generic tangent |  Two tangents |  Infinitely many tangents | Another drawing | No drawing |
| 12% | 33% | 16% | 7% | 4% | 7% |

From Vinner, 1991.

In Table 1, only 24% were willing to draw a tangent that crossed the curve, and in Table 2, only 26% would draw a tangent that crossed. In Table 3, 56% of the incorrect responses indicated line(s) that looked like tangent to circles.

The prototype phenomenon is not the only obstacle that students encounter when attempting to understand the tangent concept. Students also attend to the wrong aspects of diagrams used in explaining the tangent concept. Orton (1980) interviewed 110 students from a wide variety of high schools and colleges in England regarding their understanding of elementary calculus. Orton presented the graphic with accompanying questions (see Figure 12) to each student.

*The diagram shows a circle and a fixed point P on the circle. Lines PQ are drawn from P to points Q on the circle and are extended in both directions. Such lines across the circle are called secants, and some examples are shown in the diagram.*



(a) How many different secants could be drawn in addition to the ones already in the diagram?

(b) As Q gets closer to P, what happens to the secant?

Figure 12. A Rotating Secant Illustration. From Orton, 1984.

Orton found that "... over 40 percent of all those interviewed were unable to state that the secant became the tangent, despite considerable encouragement, through further questioning, to say more about what happened, until they ran out of things to say about the situation ... Typical responses included, 'The line gets shorter', 'It becomes a point', 'The area gets smaller', and 'It disappears'." (Orton, 1984 p. 25). It is not clear whether the students who answered incorrectly failed to acquire a mental image of a rotating secant. The typical responses quoted above are correct and consistent with an image of a rotating secant. We can only speculate that they may have had an image of a tangent but not the vocabulary to express it. It is clear, however, that the students were attending to the wrong aspects of the diagram.

Another researcher attempted to improve student understanding of the tangent concept with the use of computer programs. David Tall (1986) developed several software programs to help students develop a better understanding of calculus. *Gradient* allows the student to enter a function and a fixed point. The program then draws successive secants corresponding to a decreasing $\Delta x$. As each secant is displayed, its slope and $\Delta x$ is entered on a table adjacent to the graph. *Gradient* also allows the student to enter a value for $h$ for the function

$$G(x) = \frac{f(x+h) - f(x)}{h}.$$

The program then displays a secant line moving along the curve with a fixed $\Delta x$ $(\Delta x = h)$ and graphs the slopes of the secant lines, $y = G(x)$, simultaneously. *Magnify* plots a function and then allows the student to zoom in on the graph. The program is similar to the zoom feature on graphing calculator but displays the magnified graph simultaneously with the original graph and a rectangle around the part of the graph that is magnified.

These software programs are part of what Tall calls "generic organizers," which are environments "enabling the learner to explore examples of mathematical processes and concepts . . ." that provide experiences from which the learner can abstract "higher order concepts embodied by the organizer" (p. vii). Generic organizers, according to Tall, must be complemented with an "organizing agent," usually a teacher, that helps the students focus on the relevant attributes of the generic organizers and guides the students in picking examples to investigate with the organizers.

Tall's thesis is that a curriculum designed around mathematical structure, without taking into account the cognitive structure of the students, is not optimally effective. The structure of mathematics suggests a sequence of teaching based on definition, theorem, followed by proof. In calculus instruction, the tangent is typically introduced after the student has studied limits. This approach, based on the structure of mathematics, requires this order because the formal definition of "tangent" involves limit. However, since most calculus students do not have a solid understanding of limits, Tall suggests that the definition of tangent as a limit is lost on the students. In this study, the tangent was initially introduced to the experimental groups on the computer as a line through two points very close together on a graph. The subjects could adjust the two points on the computer to make them closer together. Later in the course, the tangent is defined as the “limiting position” of the secants, and the tangent only exists where the graph looks straight when magnified. Tall’s cognitive approach to teaching makes use of generic organizers that allow students to experiment with graphical examples and non-examples of mathematical concepts within a framework that is consistent with the students’ existing knowledge. Tall believes that calculus instruction must be consonant with the students’ existing cognitive structure. This approach, which is intended to resonate with the learners’ intuition, should prepare the students for a more rigorous understanding later. Since a visual approach is more intuitive than the analytic, these organizers emphasize the graphic approach.

Tall found that students who worked with his software, along with the support of teachers, were better able to visualize the slope of a curve and to sketch graphs of derivatives when given the graph of the function only. The experimental groups also outperformed the control groups in ability to generalize the tangent concept to non-differentiable points and to tangents that crossed curves or were coincident with graphs of functions. The experimental groups, however, were more likely to define the tangent as a line through two points that are very close together on a graph.

## Functions

An understanding of functions is central to grasping the tangent concept. Consequently, past research on the teaching and learning of functions is pertinent to the present study. The difference quotient, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, is a composition of functions that, for most students, is complicated. The most difficult aspect of the entire curriculum on functions involves functions whose inputs are functions (E. Dubinsky in personal communication reported in Yerushalmy & Schwartz, 1993). The calculus teacher sees the difference quotient as the slope of a secant line. This, in the language of Dubinsky and Harel (1992), is an encapsulation of the process conception of function to an object conception. We can view the difference quotient as a function where $\Delta x$ is the independent variable and slope of the secant line is the dependent variable. So, for example, in the function $f(x)=x^2$ (see Figure 13) at the point (-1,1) we can look at the function $g(\Delta x)=\frac{f(-1+\Delta x)-f(-1)}{\Delta x}$, or $g(\Delta x)=\textit{slope of the secant line}$. An algebraic simplification of this function is $g(\Delta x)=\Delta x-2$ for all $\Delta x \neq 0$. A table expressing this function is shown in Figure 14, and graph of this function is shown in Figure 15.

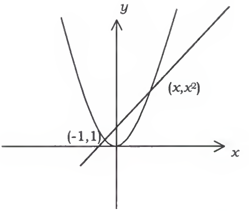


Figure 13. Graph of $y = f(x)$ with a secant line at (-1,1).

| $\Delta x$ | $g(\Delta x)$ |
|---|---|
| 4 | 2 |
| 3 | 1 |
| 2 | 0 |
| 1 | -1 |
| 0 | undefined |
| -1 | -3 |
| -2 | -4 |

Figure 14. A tabular representation of $g(\Delta x)$ = slope of the secant line.

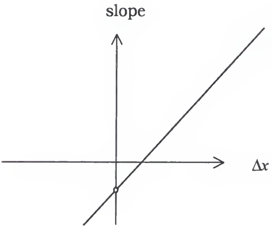


Figure 15. Graph of $g(\Delta x)$ = slope of the secant line.

The tabular and graphical representations of the slope of the secant line as a function of $\Delta x$ show a clear pattern. The student, when viewing these representations, may even be able to predict that $\lim_{\Delta x \to 0} g(\Delta x) = -2$. But none of the three representations above make salient that as $\Delta x$ approaches zero, the secant line tends toward the tangent. The rotating secant line, also a function of $\Delta x$, shows a clear pattern of the secant line approaching the tangent as Δx approaches zero. When the secant line rotates toward a limiting position as point P approaches point Q (see Figure 16), one can consider the position of the secant line as a function of the position of point P. One can also consider the position of the secant line as a function of Δx. In functional notation, we could write:

$$F(\Delta x) = \textit{Position of secant line}$$
$$G(P) = \textit{Position of secant line}$$

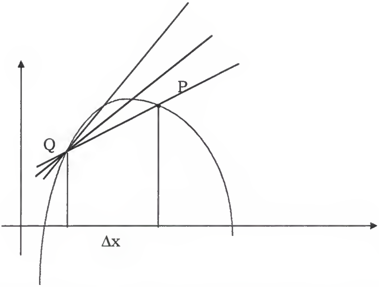


Figure 16. The position of the secant line is a function of $\Delta x$, and a function of the position of point P.

These are functions without formulas and functions that need not use numbers as input or output. These are not functions that map real numbers to real numbers, but functions wherein the domains are lengths of a line ($\Delta x$) or positions of a point (P) and ranges are positions of a line (secant). The salient features in the rotating secant image are:

> *As the line segment, $\Delta x$, decreases, the secant line rotates toward the tangent line.*
>
> *As point P moves along the graph of $f(x)$ toward point Q, the secant line rotates toward the tangent line.*

This image presents a function, not of real numbers mapped to real numbers, but of a correspondence between varying lengths of $\Delta x$ (visually, not numerically) and varying positions of a secant line (visually, not analytically). The image also presents a correspondence between position of point *P* and position of the secant line. The image of the rotating secant line represents a qualitative function rather than an analytical function. I hypothesize that the qualitative understanding of this functional relationship will benefit students when they are attempting to make sense of tangents, their slope, and the limit of the difference quotient. Many students will develop an understanding of these concepts by attending to both the imagistic and analytic aspects of the concepts.

Educational researchers have identified some common problems that students have with functions. Many students think that an explicit formula is necessary to define a functional relationship (Vinnner, 1983). Most high school students do not have a strong understanding of the relationship between symbolic and graphical solutions to problems (Dreyfus & Eisenberg, 1987). Many students have trouble understanding the connection between the intersection of graphs of two functions and the solutions to the simultaneous equations (Herscovics, 1989). Most students avoid using graphical representations of functions in problem solving (Vinner, 1989, Eisenberg, 1992). Students often confuse graphs of physical situations with pictures of the situation (Dugdale, 1993). For example, when asked to find a graph that best represents velocity verses time of a person riding a bicycle over a hill, students are most likely to select a graph that looks like a hill. Another problem is that students tend to think of functions one point at a time and lack across-time understanding (Monk, 1992).

Monk described two types of understanding about functions: pointwise and across-time. A pointwise understanding is one in which the student is able to determine and interpret specific values of functions. An across-time understanding allows the student to see patterns of change in functions. For example, a pointwise understanding allows the student to plug $t = 2$ into an equation for velocity, $v = t^2 + 32$, and interpret this to mean that the speed of an object is 36 meters per second after an object has traveled for 2 seconds. However, a student who is limited to a pointwise understanding of functions would not know that the object is speeding up as time increases – this would require an across-time understanding. Another example comes from a study by Monk (1988) in which calculus course graduates were presented a graph similar to that in Figure 17 and asked the following questions:

$A(p)$ = area of the region bounded by $y = f(x)$, the $x$-axis, the $y$-axis, and $x = p$

1. Determine the values of $A(1)$ and $A(3)$.
2. The point $p$ moves from 4.5 to 6.0. Does the area $A(p)$ increase or decrease?

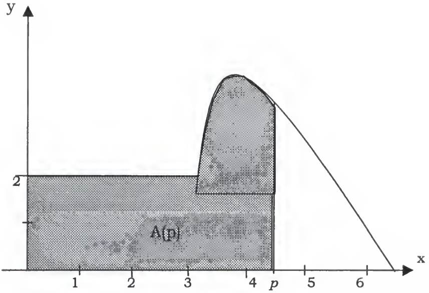


Figure 17. A(p) is equal to the area of the shaded region.

Eighty-five percent of the subjects were able to answer the pointwise question (1), but only 53 percent correctly answered the across-time question (2).

Several researchers have adopted a common terminology, process and object conceptions, in describing student learning about functions (Dubinsky & Harel, 1992, Moschkovich, Schoenfeld, Arcave, 1993, Kieran, 1993, Sfard, 1992, Schwingendorf, Hawks & Beineke, 1992). A function can be thought of as a process of associating elements of a domain with elements of a range: this is a process conception. One has an object conception of functions when he or she views functions as things to which actions can be applied. As a student progresses from a process conception to an object conception, his or her understanding becomes more abstract and generalized. Sfard (1992) called a process conception "operational" and the object conception "structural." One's view of functions is not clearly operational or structural, but more-or-less operational or more-or-less procedural. Students with a strong object conception of functions tend think of them as entities that can be acted upon and categorized into sets. Hence, actions such as translating or differentiating can be applied to functions; functions can also be categorized into sets such as transcendental or algebraic functions. Selden and Selden (1992) posit that an object conception of functions is necessary for understanding the main ideas of calculus. In order for a student's conception for differentiation to progress beyond a mere mechanical understanding, he or she must think of functions as objects to which the process of differentiation is applied. Most calculus students do not treat functions as objects when differentiating; they perform these actions mechanically and are simply mimicking the behavior of their teachers (Selden & Selden, 1992).

Moschkovich, Schoenfeld, and Arcavi (1993) adopted the process-object terminology in a study at the Functions Group at the University of California, Berkeley. They hypothesized that a multidimensional level of thinking is required for a successful understanding of functions. That is, students must have both a process and object conception of functions. Successful learners must also possess a tabular, algebraic, and graphical conception of functions. Furthermore, students must be able to fluently change perspective among (1) the different representations of functions and (2) a process or object conception of functions. The researchers suggest that students who progress from a process conception to object conception of functions do not always deal with functions as objects. Instead, successful students move back and forth between an object conception and process conception. The symbolic representation of functions emphasizes their process character. The graphical representation of functions de-emphasized the process characteristic while making functions more like objects. So Moschkovich et al. are suggesting that successful learners alternate between the graphical and analytical representations of functions. This hypothesis is consonant with the V/A theory. I propose that students who do not see the rotating secant line will be deprived of an important visual aid that facilitates progressing from a process conception of this function to an object conception.

Sfard (1992), in agreement with Moschkovich et al., posits that effective problem solving usually involves alternating between process and object conceptions of mathematical ideas. Furthermore, students are more inclined to think of functions as objects when looking at a graph and more likely to conceive of functions as processes while computing values. Since Sfard also posits that mental images can serve the same purpose as pictures in view, we can reasonably suggest that the mental image of the rotating secant will help students make sense of tangents and their analytical representations.

Dubinsky and Harel (1992) expanded the process-object terminology use in describing students' understanding of functions to include "action conception." A person performs an action when he or she transforms some mathematical object using a specific algorithm or formula. The action conception is a pre-process conception that is limited to plugging numbers into a formula to get an output. Eisenberg and Dreyfus (1994) suggest that the action conception is a static view (point by point) while the process conception is dynamic. When reporting on students' learning about transformations of functions, the researchers suggested that students would only have a complete understanding of transformations of functions if they have achieved an object (process) conception of functions. This necessitates that the students have a dynamic view of transformations rather than a static view. Students with a static view of transformations considered only the original and final positions or shapes of graphs. Other students, with a process conception of functions, viewed the transformations as a dynamic process of sliding, shrinking, or twisting. The student with a dynamic view of transformations has an advantage over the student with a static view because the dynamic view allows one to see all the points of the original function moving to a new location. The static view of transformations only sees a before and after--a mechanistic understanding where the student knows, for example, to place the graph of $g(x) = x^2 + 1$ up one unit from the graph of $f(x) = x^2$ because the teacher "said so." The knowledge that the graph moves up because one unit is added to each ordinate on the graph of $y = f(x)$ is facilitated by a dynamic view of the transformation. Many students are unable to see the dynamic action from the initial and final states of the transformation. The researchers concluded that media such as film or computer animations might help students develop a dynamic view of transformations.

Eisenberg (1992) argued that students need to develop a sense for functions that is similar to a number sense. In order for students to have function sense, they must be proficient at making connections between analytic and graphical representations of functions. For example, students with a high level of function sense are able to draw graphs of fourth and fifth degree polynomials. One with function sense can use the visual and analytic aspects of functions to quickly see that $x^2 - \sin(x) + 1 = 0$ has no real solutions and $\sin(x) = x$ has exactly one solution. These competencies require that the students make strong connections between the analytic and graphical representations of functions. (Again, this is consistent with the VA theory.) Most college students cannot solve these problems (Dreyfus & Eisenberg, 1990). The problem is caused primarily by the students' avoidance of the visual aspect of functions. Just as with number sense, good function sense requires flexibility in thinking, a propensity to view functions from multiple perspectives, and the ability to choose the representation most appropriate for the problem.

Researchers have confirmed that differing instructional treatments can affect student's conception of functions. In a study by Schwingendorf, Hawks, & Beineke (1992), first-year calculus students who worked with the computer program *ISETL*, software designed for teaching mathematics, were better able to connect graphical and analytical representations of functions. Moschkovich et al. (1993) found that the software Grapher helped students develop an object conception of functions and graphs. A 1990 study by Dunham (as cited in Demana, Schoen, & Waits, 1993) demonstrated that the use of graphing calculators in a college algebra class can improve females' performance on graphing items in a posttest. Breidenbach et al. (1992) showed that having students write computer programs for functions facilitated the students' move toward an object conception of function. Dugdale (1989) found that students who worked with *Green Globs* software (Dugdale & Kibbey, 1983) and were encouraged to make conjectures about functions based on visual considerations developed a more qualitative view of functions. Dugdale (1993) contends that student use of graphing software such as *Green Globs* facilitates students' progress toward viewing functions globally instead of one point at a time. Eisenberg and Dreyfus (1994), by using the computer to emphasize transformations of classes of functions, were able to improve students' understanding of the connection between algebraic and graphical representations of functions. Monk (1992) found that, when the students were allowed to work with physical models, almost all the subjects in his study were able to answer across-time questions about the functional relationship of the height of the top of a ladder as the bottom is pulled away from a wall.

The rotating secant line describes several functional relationships, so a student's conception of functions will affect his or her understanding of tangent – the limiting position of the secant. Eisenberg described a type of understanding, function sense, which is necessary for learning major calculus ideas. Students need to stress both the graphical and analytical aspects of functions in order to develop a strong sense for functions. Dubinsky et al. have described a type of understanding, object conception of function, that is necessary to achieve this function sense. Moschkovich et al. (1993) maintained that students must move between representations and conceptions of functions. Moses suggested that an across-time understanding of functions is required for successful calculus students. Theories on understanding functions (process-object, concept image, functions sense, across time understanding) all have a common element: The student must be able to move fluently between graphical and algebraic representations of functions.

Several researchers have demonstrated that, with the use of new technologies, we can devise instructional treatments that improve the students understanding of functions. Monk showed that a concrete representation of a dynamic function improved the students' across-time understanding of that function. This is consistent with my view that a more concrete representation of a function (the rotating secant function) can have a salutary effect on student understanding.

## Limits

Finding the slope of a tangent is a limiting process. The formal definition of $\lim_{x \to a} f(x) = L$ is

For all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$ .

This formal definition of limit is usually not covered extensively in beginning calculus; limits are generally covered from first an intuitive approach involving finding limits by examining the graphs of functions. For example, the students are asked to find $\lim_{x \to 0} f(x)$ in Figure 18, or $\lim_{x \to 0^-} g(x)$ in Figure 19.

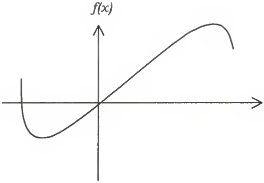


Figure 18. Students are asked to find $\lim_{x \to 0} f(x)$.

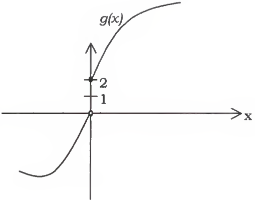


Figure 19. Students are asked to find $\lim_{x \to 0^-} g(x)$.

Textbooks also ask students to find limits such as $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ by constructing a spreadsheet similar to the one in Figure 20. The student should see that, although $\frac{x^2 - 4}{x - 2}$ is undefined at $x = 2$, he or she will see that as $x$ approaches two from both the left and the right, the function approaches four. As the student experiments with the spreadsheet by choosing numbers closer and closer to two, he or she may see that one may make the function as close to four as one likes by making $x$ close enough to two. While this is very close to the formal definition of limit, it also gives the student an intuitive understanding of limits.

| x | 0.000 | 1 | 1.5 | 1.7 | 1.9 | 1.99 | 1.999 | 1.9999 | 2 | 2.0001 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (x^2-4)/(x-2) | 2.000 | 3.000 | 3.500 | 3.700 | 3.900 | 3.990 | 3.999 | 4.000 | #DIV/0! | 4.000 | 4.001 | 4.010 | 4.100 |

Figure 20. Students may construct spreadsheets in order to develop an intuitive understanding of limits (spreadsheet prints x^2 for $x^2$).

The body of research on this subject is divided between limits of sequences and series, and limits of functions. The research conducted so far has reached one conclusion consistently: Students have a poor understanding of the formal definition of limit (Davis & Vinner, 1986, Tall & Vinner, 1981, Cornu, 1986, Cottrill et al., 1996).

Sierpinska (1987, 1990) attributes student's problems with understanding limits to the student's static view of functions. Many calculus students can only deal with one calculation at a time (an action conception of functions) and are unable to think of function changing as the domain changes (a process conception of functions). Cottrill et al. (1996) hypothesize that students fail to understand limits because they are unable to coordinate the mental processes of seeing a function approach a value in the range as the independent variable approaches another value in the domain. Davis and Vinner (1986) suggest that the language of limits causes many difficulties that students encounter. For example, the word "limit" may connote an upper bound, as in "speed limit."

An understanding of the formal definition of limit is not required to grasp the concept of tangent. David Tall (1986) concluded from previous research that because students have so much trouble with limits, a formal treatment of the topic should not be included in beginning calculus. Tall showed that students can develop a solid understanding of tangents without learning the formal definition of the limit of a function.

## Animation and Directed Attention

In mathematics textbooks, static graphics with accompanying verbiage to prompt the reader to imagine dynamic images are limited in the ability to direct attention to the pertinent details of the image. Even if the reader is successful in imagining the motion described by the text, he or she may not attend to the pertinent details. In a review of the research on the effectiveness of animated visuals in computer based instruction, Lloyd Rieber (1990) concluded that learners inexperienced in a subject may not attend to the relevant details in an animation. However, in a well-designed computer animation of an action metaphor, the author can include prompts that direct the student to the important parts of the images. For example, in the animation of the rotating secant line, $\Delta x$ and $\Delta y$ can be emphasized with thickened lines.

The viewers attention can be even more strongly directed by allowing him or her to control a pertinent feature of the animation with the mouse. Indeed, Reiber also concluded from his review that the interactive capability of the computer would likely allow for the most effective use of animation. In a study investigating the effectiveness of animation on improving students' estimates to problems involving motions, Reed (1985) concluded that novices often attend to the wrong features when simply viewing animation. However, adding interactive features to the animation program can ameliorate this problem.

## Kinaesthetic Imagery

Imagery can be of a kinesthetic nature as well as a visual one. Try to answer the following question: When your car enters into a typical intersection where you plan to make a left turn, through how many rotations do you typically rotate the steering wheel? When I have asked this question, the actions of the respondent almost invariably indicate a kinesthetic approach to answering the query. Most respondents would gaze ahead, place one or two hands on an imaginary steering wheel, and then rotate this imaginary wheel by turning it in a counterclockwise rotation. The respondents typically did not look at their hands to determine the number of rotations, which demonstrated a kinesthetic/analytic approach. Also, the respondents gazed forward. I submit that they were imagining a typical intersection and remembering how many turns by attempting coordination of visual and muscle memory. They used a schema for turning a car that involved a synthesis of visual and kinesthetic memory.

We can experience mathematics with our tactile as well as our visual sense. When discussing interactive dynamic visualization software such as *Geometer's Sketchpad* (Jackiw, 1991), Goldenberg et al. (1992) used the word "feel" to describe one's interaction with the computer mouse and the elements of the visual display. "You can feel the relationship between variable and image, much as you feel the relationship between your actions upon a radio dial and the station that you tune in" (p. 244). Those who have tuned in analog radios can appreciate this. But it is also worth noting that the word "feel" pertains to both perception through the sense of touch and to believe or think (as in "I feel that these variables are somehow related.").

An example of computer software that allowed students to utilize both visual and kinaesthetic imagery is *GRID Algebra,* a program in which students build up algebraic expressions using a mouse to drag polynomials into cells of a grid – movement to the right and left corresponds to addition and subtraction; movement up and down corresponds to multiplication by a whole number. Hewitt (1995) studied subjects using GRID Algebra and concluded that students using the program developed not only a visual image of building algebraic expression, but also a kinaesthetic imaged related to movements of the mouse. When simplifying an expression such as $\frac{3(x+4)}{2}-5$, the students associated the operations of addition, subtraction, multiplication, and division with movements of the mouse and the resulting visual display on the screen. The synthesis of the two types of imagery, visual and kinaesthetic, helped the subjects remember the arbitrary rules for interpreting algebraic expressions. Furthermore, the students developed a better understanding of the concept of inverse operations because addition and subtraction were associated with right and left movements; division and multiplication were associated with up and down movements. One can conjecture that the students understand readily that up is the inverse of down, and left is the inverse of right. Connecting the abstract concepts of inverse properties of arithmetic operations with the more concrete inverse nature of movements likely helped the students understand.

Seymour Papert (1980), when reflecting on how his childhood fascination with gears helped him to learn about mathematics, stated that the gear connects "with the formal knowledge of mathematics, it also connects with the 'body knowledge,' the sensorimotor schemata of a child. You can be the gear, by projecting yourself into its place and turning with it. It is this double relationship – both abstract and sensory – that gives the gear the power to carry powerful mathematics to the mind" (p. viii). Howard Gardner (1993) observed that a person best understood how an object works when utilizing a "fusion of bodily and spatial intelligences" (p. 232).

## Conclusion

If mathematical thinking involves an interplay between visual and analytic reasoning, as Zaskus et al. suggest, students who acquire the mental image of the rotating secant should best understand the tangent concept. The students must first generate the image, then study the image in accordance with the VA theory by alternating their thinking between the visual aspects and the analytic aspects of the concept. Dynamic mental images are difficult for many students to develop because moving images are difficult to communicate. There are two avenues for helping students acquire such a mental image: (1) by improving their imagery ability and (2) by providing a more explicit description of the image. In this investigation, the effectiveness of the latter approach was studied: the effect of visualization prosthetics was tested by having students make predictions about a dynamic image. Previous studies have examined whether visual-spatial skills can be taught and whether these skills are correlated to mathematics learning. The objectives of this study are to determine whether spatial skill can be augmented and whether an acquired image can improve student understanding of an abstract mathematical concept.

The constructs of concept definition and concept image (Tall & Vinner, 1981) provide a basis for determining the success with which a student acquires a concept. The concept image consists of all the impressions, mental images, and memories of past experiences of the student that are evoked by the concept name. The concept definition refers to the formal definition of a mathematical concept. A potential conflict factor is something in the concept image that is inconsistent with other parts of the concept image or with the concept definition. Potential conflict factors will not cause a problem for the student unless they are elicited. For example, the belief that a tangent will touch but not cross a curve is inconsistent with the concept definition of "tangent." The potential conflict factor may not cause a problem for the student until he or she attempts to draw a tangent at an inflection point. When the student is presented with a task that evokes this conflict, the potential conflict factor will be called a cognitive conflict factor. This study will attempt to evaluate the students' concept image of the tangent concept by eliciting potential conflict factors to see how successful the students are at resolving these conflicts with the concept definition. The measure of how well a student acquires the tangent concept will be determined by how closely his or her concept image is consonant with the concept definition.

Visual imagery is recognized as an important component of mathematics learning and problem solving. Students vary in both their propensities to visualize as well as their ability form mental visual images. Numerous studies have demonstrated that spatial ability is positively correlated to mathematical achievement. It is possible that spatial ability is necessary to form mental images necessary for understanding mathematics concepts. Investigating the effect of differing presentations of images may shed some light on the ways that spatial visualization supports mathematics learning. What role does spatial ability play in mathematics learning? Is spatial ability related to acquisition of dynamic mental images? Can limitations in one's spatial ability be overcome by different instructional treatments?

Both Lean and Clements and Presmeg found that students who preferred the analytic mode outperformed the students who preferred the visual mode. This might lead one to conclude that the analytic mode is superior for solving mathematics problems. But it is interesting that Lean and Clements (1981) also found that students who preferred the verbal-logical approaches to mathematics outperformed the more visual students on tests of spatial ability. Since mathematics is a discipline that deals in abstractions, it makes sense that those most capable of analytic thought are likely to perform best on mathematical tasks. It also makes sense that those least capable of abstract thought are likely to prefer the more concrete visual mode. But this is consistent with the VA theory since it posits that both modes of thought are required for mathematical processing.

David Tall was able to improve student understanding of the tangent through the use of a “generic organizational system” (Tall, 1986, p. 34) that required the use of a generic organizer (computer programs “*Gradient* and *Magnify*”) and an organizing agent (the teacher) who helped direct each student’s attention to the relevant aspects of the organizer. The approach to teaching calculus was modified to remain consistent with the cognitive structure of the students rather than the logical structure of mathematics. The tangent was introduced as a line through two very close points on a graph, not as a limit. While Tall’s approach was effective, this study leaves open several questions: What aspects of the program contributed to the improved understanding of the students? Does the generic organizer require an organizing agent? Was the improvement due to the generic organizational system or to the cognitive approach to teaching the concepts? Is the complete system necessary for improvement?

Students view, manipulate, and analyze visual mental images in order to solve mathematical problems and make sense of mathematical ideas. Because freshman calculus is a study of change, dynamic images are likely to help in understanding most calculus ideas. The VA theory postulates that both visual and analytic thinking are necessary in order for students to develop a solid understanding of mathematical concepts. In cases of action metaphors, where the images are dynamic, students are often deprived of the visualization because they are unable to translate a linguistic description into the appropriate dynamic image. I posit that a more concrete depiction of the dynamic image will enable the student to acquire the mental picture required to have access to the visualization side of the VA triangle depicted in Figure 6. But seeing does not mean understanding. The viewer must attend to the relevant aspects of the image. I also posit that the interactive element of dynamic images presentations, controlling the image with the mouse, can be designed to draw the viewer's attention to the important aspects of the animation and help the viewer become more engaged in the picture. The kinesthetic component of this interactivity can further enhance learning by marshaling a multisensory approach to learning.

The rotating secant line provides a visual image that describes several functional relationships, so a student's conception of functions will impact his or her understanding of tangent – the limiting position of the rotating secant line. While most calculus students have a pointwise understanding of functions, it is the across-time understanding that is most useful in understanding most ideas in calculus. The pointwise view of functions is static, non-visual and process orientated. The across-time view is dynamic, visual, and object orientated. Another obstacle to understanding functions is that many students fail to make connections between algebraic and visual representations of functions. This failure severely limits one's ability to make sense of the difference quotient. Past research has demonstrated that different instructional treatments can improve the students' conception of functions. In many cases, the instructional environments that most helped students involved the use of new technology such as graphing calculators or computer software. The present study will attempt to determine if new computer software can help students acquire the image of a rotating secant line. I hypothesize that the image of a rotating secant line will assist the student in achieving an across-time understanding of the difference quotient and make connections between its visual and analytic representations.

# CHAPTER 3
# METHODOLOGY AND HYPOTHESES

The purpose of this study was to compare the effects of static, animated, and interactive-animated computer lessons on students' acquisition and schematization of mental visual images, investigate the correlation between the accuracy of the students' rotating secant image and their acquisition of the tangent concept, and identify any interactions between gender and presentation type.

## Subjects

Students in this study were enrolled in the first semester of a three-semester calculus sequence at Shippensburg University. Most of the students at this midsize university in rural Pennsylvania were from Pennsylvania. Students from all five sections of Calculus I participated in the study. Two students did not consent to entering the study and were excluded. Most of the students in these classes were majoring in chemistry, mathematics, secondary mathematics education, physics, or biology. All the subjects were undergraduate students. They were mostly first and second year students who had taken precalculus. There were 40 males and 42 females.

## Lesson Content and Versions

A computer-based lesson, developed by the researcher with Toolbook Instructor II (Asymetrix, 1996) software first reviewed functional notation and the concept of slope. Then the secant line and its slope were introduced with several screens of static graphics and textual material. The concept of tangent was then presented as the limiting position of the secant lines as $\Delta x$ approaches zero. This screen was prepared in three different versions.

1. Static: a static graphic of several secant lines and a tangent line is accompanied by text prompting the student to imagine how a secant line rotates to the limiting position of tangent as $\Delta x$ approaches zero.
2. Animated: an animated graphic of a rotating secant line and decreasing $\Delta x$ is accompanied by text pointing out that the secant rotates to the limiting position of tangent as $\Delta x$ approaches zero. A thickened line is used to highlight $\Delta x$. The screen has a "repeat" button that will allow the subject to view the animation as often as desired.
3. Interactive-animated: an animated graphic of the rotating secant line and resulting tangent is controlled by the subject by dragging one end of $\Delta x$ (either increasing or decreasing $\Delta x$) with the mouse.

This graphic, accompanied by text that instructs the student how to operate the animation, points out that the secant rotates to the limiting position of tangent as $\Delta x$ approaches zero. A thickened line is used to highlight $\Delta x$.

## Measures

The Card Rotations Test (Ekstrom et al., 1976) provided a measure of the student's ability to visualize in two dimensions. Each question in this test presented a reference figure followed by eight isomorphisims of the figure that had been either rotated in the plane or flipped and rotated. Within each set, students were required to distinguish between figures that were "same" (merely rotated) or "different" (flipped and rotated). In each part of this two-part test, students were allowed three minutes to identify eighty figures in ten sets of eight. The score was calculated by subtracting the number of incorrect responses from the number of correct responses.

Spatial visualization is involved in tasks that require mentally rotating or moving visual images. Tartre (1990) suggested that the Card Rotations test (Ekstrom et al., 1976), the Form Boards test (Ekstrom et al., 1976) and the Space Relations section of the Differential Aptitude Test (Bennett, Seashore, & Wesman, 1973) are measures of spatial visualization ability. Subjects in the present study will be required to mentally rotate secant lines and predict changes that result from these rotations on a static diagram. Because spatial visualization is required to perform these tasks, the Card Rotations test was used as a pretest in this study to determine its usefulness as a covariate.

The constructs of image acquisition (IA) and concept acquisition (CA) were measured by a posttest. IA was a measure based on the accuracy with which the subject, when presented with a static graphic of a secant line, can predict the outcomes of animating the picture. CA was a measure of the subject's ability to draw tangents to graphs in novel positions and identify connections between analytical and graphical representations of tangents. The posttest was revised after the pilot study. Both the IA score and the CA score was calculated by count the number of correct responses. It was anticipated that the Card Rotations Test would serve as a covariate, and that the scores on the post-test would be adjusted according to the scores on the pretest.

Table 5. Question Numbers for the Image Acquisition and Concept Acquisition Posttest.

| Construct | Question Numbers |
|---|---|
| Image Acquisition (IA) | 1-14 |
| Concept Acquisition (CA) | 15-24 |

## Reliability and Validity

The posttest, while administered as a single test, was treated as two posttests designed to measure two different constructs – image acquisition and concept acquisition. Content validity was analyzed by four experts – two calculus professors, one mathematics education professor, and one educational psychologist. The internal consistency reliability of each test was calculated using the Kuder-Richardson Formula 20. The image acquisition test has a KR-20 score of .71 and the concept acquisition test had a KR-20 score of .52. KR-20 scores in this range are acceptable by presents educational research standards. For example, Battista, Wheatley, and Talsma (1982), in a well-respected study on spatial visualization and cognitive development that was reported in the Journal for Research in Mathematics Education, used a test for cognitive development that had a KR-20 score of .54.

It may have been possible to increase the reliabilities of the tests in the present study by increasing the number of test questions. However, increasing the number of problems on the tests would have increased the time required for the students to complete the tests. Some of the cooperating instructors may have objected to taking up too much valuable class time with testing. According to Nunnally, the KR-20 scores for the image acquisition and concept acquisition tests were sufficient for basic research.

## Pilot

The software and the posttest were pilot-tested at the University of Colorado at Colorado Springs. The evening calculus class met in the computer lab and took the posttest after working though the CBI lesson. Both the researcher and the class instructor took time after the completion the test to discuss both the program and the tests with the students. As a result of this pilot test, the software was modified to include more complete review of the connection between functional notation and graphical representations of functions. The navigation was also changed to allow students to move backward in the program. Also as a result of the pilot test, the researcher eliminated or re-worded five questions from the posttest and corrected two typographical errors. The pilot test also helped the researcher determine the length of time to allow the students to work through the software and complete the posttest.

## Procedure

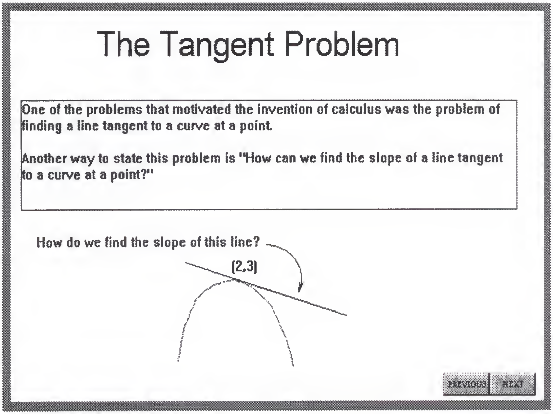
The students in each class were randomly divided into three groups: static graphic, animated graphic, and interactive-animated graphic. Random assignment was achieved by numbering the names on class rosters 1,2, or 3 sequentially until each name was assigned a number. Names with number 1 were assigned to the interactive-animated group, number 2 the animated group, and number 3 the static group. Students participated in this study early in the semester, before their instructor introduced the tangent concept, but after the instructor introduced the limit concept. The lesson and data collection were completed in two class periods. On the first day, the researcher distributed and read aloud the consent form, then administered the Card Rotation Test. At the next meeting, the students were allowed 25 minutes to go through the computer lessons on the tangent. This was followed by a fifteen-minute posttest.

## The CBI Lesson

The objective of the CBI lesson was to impart an understanding of the formal definition of “tangent” by first describing a visual representation of the tangent as the limiting position of a rotating secant, then presenting the analytic definition of tangent as a line with a point common to the graph of the function with slope that is the limit of the difference quotient. The formal definition of a tangent to the graph of $y = f(x)$ at point $(a,b)$ is the line through $(a,b)$ with slope

$$m = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

The student must have an understanding of the concepts of slope, limits, and functional notation in order to apprehend the formal definition of “tangent”. The CBI lesson began with a statement of the tangent problem (see Figure 21) followed by an overview of the program.



The Tangent Problem

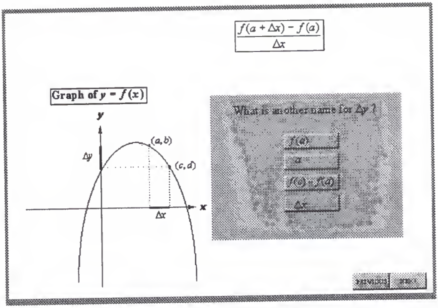
One of the problems that motivated the invention of calculus was the problem of finding a line tangent to a curve at a point.

Another way to state this problem is "How can we find the slope of a line tangent to a curve at a point?"

How do we find the slope of this line?

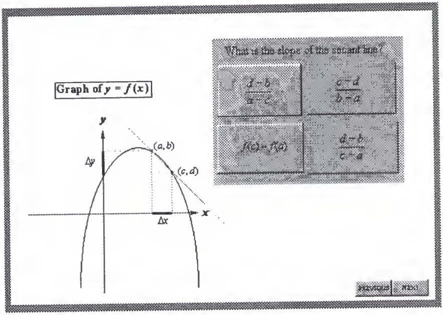
Figure 21. Statement of the tangent problem.

The statement of the tangent problem was followed by a review of slope and functional notation. Students were then directed to incorporate the notions of slope and functional notation into the components of the difference quotient $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. For example, they were asked to identify the various ways to state the slope of the secant line through points $(c,d)$ and $(c+\Delta x, f(c+\Delta x))$ (see Figures 22, 23, 24, 25).



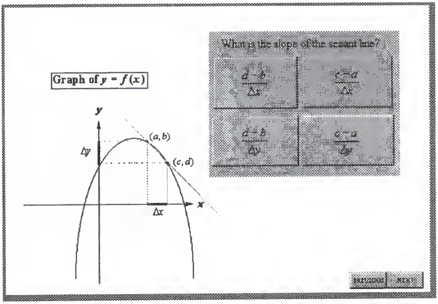
What is another name for $\Delta y$ ?

Figure 22. The student is asked to restate $\Delta y$.



What is the slope of the secant line?

Figure 23. The student is asked to find the slope of the secant line.



What is the slope of the secant line?

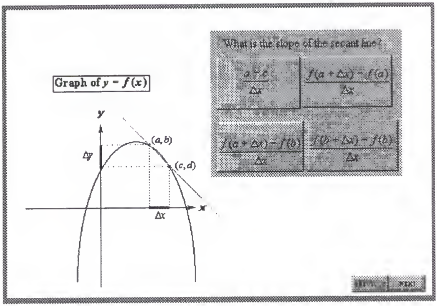
$\frac{d-b}{\Delta x}$

$\frac{c-a}{\Delta x}$

$\frac{d-b}{\Delta y}$

$\frac{c-a}{\Delta y}$

Figure 24. The student is asked to restate the slope of the secant line.



What is the slope of the secant line?

$\frac{a-c}{\Delta x}$

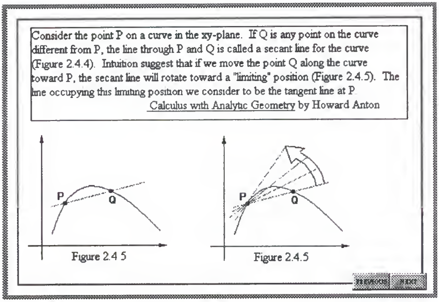
$\frac{f(a+\Delta x)-f(a)}{\Delta x}$

$\frac{f(a+\Delta x)-f(b)}{\Delta x}$

$\frac{f(b+\Delta x)-f(b)}{\Delta x}$

Figure 25. The student is asked to restate the slope of the secant line in terms of the formal definition of tangent.

The student must have more than an understanding of slope, functional notation, and limits in order to grasp the formal definition of the tangent. A complete concept image includes all images associated with the definition. The image of the rotating secant line, an essential component of the concept image of tangent, was presented in the CBI lessons in three different versions. The static version was presented as a typical textbook exposition with a graphic where the reader is prompted to imagine the rotation of a secant line that would take place if one point of intersection of the secant and the graph of a function (point Q) moved closer to the other point of intersection (point P) (see Figure 26). The animated version provided a moving picture of a rotating secant line as point Q approached point P. This version also highlighted the diminishing $\Delta x$ with a thickened blue line. The students could view the animation as they wished by clicking on the Show Me button (see Figure 27). The interactive animated version allowed the student to control the movement of the secant line by decreasing $\Delta x$ (moving the mouse to the left along the thickened blue line) or by increasing $\Delta x$ (moving the mouse to the right along the thickened blue line) (see Figure 28).



Consider the point P on a curve in the xy-plane. If Q is any point on the curve different from P, the line through P and Q is called a secant line for the curve (Figure 2.4.4). Intuition suggest that if we move the point Q along the curve toward P, the secant line will rotate toward a "limiting" position (Figure 2.4.5). The line occupying this limiting position we consider to be the tangent line at P.

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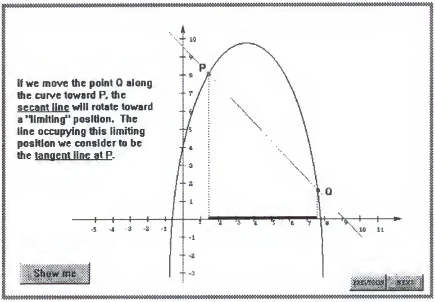
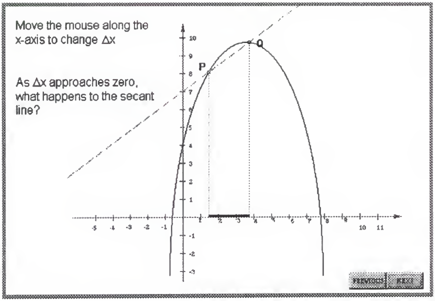
Figure 26. The static version prompted the student to imagine movement of the secant line as point Q approached point P.



If we move the point Q along the curve toward P, the secant line will rotate toward a "limiting" position. The line occupying this limiting position we consider to be the tangent line at P.

Figure 27. The animated version displayed a motion picture of the secant line rotating as $\Delta x$ diminished.



Move the mouse along the x-axis to change Δx

As Δx approaches zero, what happens to the secant line?

Figure 28. The interactive animated version allowed the student to control the movement of point Q and the resulting rotation of the secant line by controlling the size of $\Delta x$ with the mouse.

## Design

A 3 (treatment)X 2(gender) ANCOVA was used to compare the groups. The hypotheses were tested using analysis of covariance with the pretest score as the covariate. The independent variables were graphic type and gender. The dependent variables were the scores on the IA and CA portions of the posttest. After it was determined that the correlation of pretest scores with image acquisition scores was not statistically significant, the data were analyzed using an ANOVA. Because the ANOVA revealed no statistically significant interaction between gender and treatment, the interaction term was eliminated from the model and the ANOVA was re-run. The Tukey HSD would serve in as a post hoc test on any differences among groups.

Table 6. Treatment by Gender Matrix.

| | Static Graphic | Animated Graphic | Interactive Animated Graphic |
|---|---|---|---|
| male | group 1 | group 2 | group 3 |
| female | group 4 | group 5 | group 6 |

This study was designed to answer the following questions:

1. Is there a treatment effect on IA or CA?
2. What is the correlation between IA and CA?
3. Is there an interaction between gender and treatment?

## Hypotheses

1. The animated group will outperform the static group in image acquisition.
2. The interactive-animated group will outperform the animated group in image acquisition.
3. Females will benefit more than males from animated presentations in image acquisition.
4. Females will benefit more than males from the interactive-animated presentation in image acquisition.

5. There will be a significant positive correlation between image acquisitions and concept formation.

# CHAPTER 4
# RESULTS

## Descriptive Statistics

The sample consisted of all students enrolled in Calculus I during Spring Semester 1998 at Shippensburg University. Two outliers were identified in the pretest scores. The subjects that scored the outliers either did not understand the instructions for the test or did not take the test seriously. Consequently, all scores from these two subjects were omitted from the data analysis.

The means and standard deviations for the pretest and the posttests are given in Tables 7 through 14.

Table 7. Means and Standard Deviations of Groups for the Pretest

| Group | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Group 1 Interactive | 30 | 108.7 | 33.5 |
| Group 2 Animated | 24 | 112.7 | 27.1 |
| Group 3 Static | 28 | 111.9 | 19.8 |

Table 8. Means and Standard Deviations of Males and Females for the Pretest

| Gender | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Male | 40 | 114.2 | 24.6 |
| Female | 42 | 108.0 | 29.6 |

Table 9. Means and Standard Deviations of Groups for the Image Acquisition Test

| Group | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Group 1 Interactive | 29 | 9.21 | 2.14 |
| Group 2 Animated | 23 | 9.91 | 2.39 |
| Group 3 Static | 28 | 9.29 | 2.24 |

Table 10. Means and Standard Deviations of Males and Females for the Image Acquisition Test

| Gender | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Male | 39 | 9.92 | 2.65 |
| Female | 41 | 8.97 | 1.68 |
| Total | 80 | 9.44 | 2.24 |

Table 11. Means and Standard Deviations of Males and Females, by Group, for the Image Acquisition Test

| | Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Group 1 Interactive | | | |
| Males | 15 | 9.67 | 2.44 |
| Females | 14 | 8.71 | 1.73 |
| Group 2 Animated | | | |
| Males | 10 | 10.40 | 3.13 |
| Females | 13 | 9.54 | 1.66 |
| Group 3 Static | | | |
| Males | 14 | 9.86 | 2.66 |
| Females | 14 | 8.71 | 1.64 |

Table 12. Means and Standard Deviations of Groups for the Concept Acquisition Test

| Group | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Group 1 Interactive | 29 | 5.21 | 2.06 |
| Group 2 Animated | 23 | 4.52 | 1.65 |
| Group 3 Static | 28 | 3.86 | 1.67 |

Table 13. Means and Standard Deviations of Males and Females for the Concept Acquisition Test

| Gender | Number of Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Male | 39 | 5.03 | 1.98 |
| Female | 41 | 4.07 | 1.68 |
| Total | 80 | 4.54 | 1.88 |

Table 14. Means and Standard Deviations of Males and Females, by Group, for the Concept Acquisition Test

| Group | Subjects | Mean | Standard Deviation |
|---|---|---|---|
| Group 1<br>Animated | | | |
| Males | 15 | 5.67 | 2.35 |
| Females | 14 | 4.71 | 1.64 |
| Group 2<br>Interactive | | | |
| Males | 10 | 5.10 | 1.29 |
| Females | 13 | 4.08 | 1.65 |
| Group 3<br>Static | | | |
| Males | 14 | 4.29 | 1.82 |
| Females | 14 | 3.43 | 1.45 |

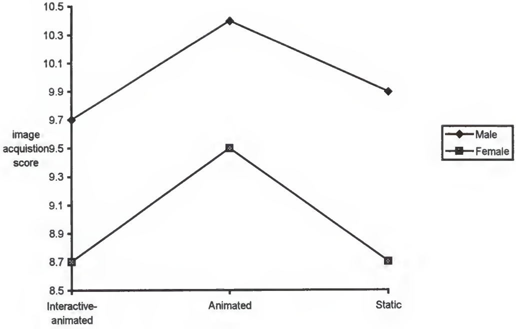


Figure 29. Graphs of the mean image acquisition scores, by group, for males and females indicate no interaction between treatment and gender.

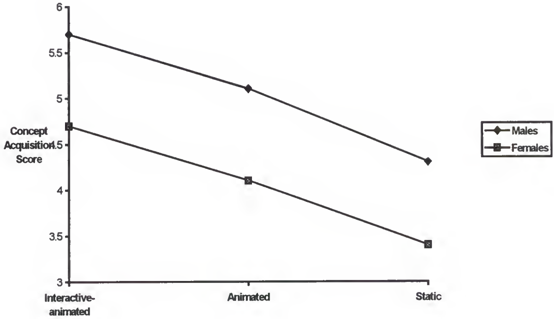


Figure 30. Graphs of the mean concept acquisition scores, by group, for males and females indicate no interaction between treatment and gender.

## Statistical Analysis

There were no significant differences among groups 1, 2 or 3 on the pretest means.

There were no significant differences among groups on the image acquisition test (see Table 15). Therefore, the following null hypotheses could not be rejected.

> The animated group will not outperform the static group in image acquisition.
>
> The interactive-animated group will not outperform the animated group in image acquisition.
>
> The interactive-animated group will not outperform the animated group in image acquisition.

Table 15. ANCOVA for Image Acquisition

| Source | DF | SS | Mean Square | F Ratio | p |
|---|---|---|---|---|---|
| Pretest | 1 | 13.70 | 13.70 | 2.80 | 0.098 |
| Gender | 1 | 14.49 | 14.49 | 2.97 | 0.089 |
| Group | 2 | 6.82 | 3.41 | .70 | 0.501 |
| Gender*Group | 2 | .55 | .27 | .06 | 0.946 |
| Model | 6 | 40.99 | 6.83 | 1.40 | 0.227 |
| Error | 73 | 356.69 | 4.89 | | |

There were no significant interactions between gender and presentation type on image acquisition (see Table 15 and Figure 29). Therefore the following null hypotheses could not be rejected.

> Females will not benefit more than males from animated presentations in image acquisition.

> Females will not benefit more than males from the interactive animated presentation in image acquisition.

The correlation between image acquisition score and concept acquisition scores was positive with a Pearson Correlation Coefficient of .165, but not statistically significant (p = .139). Therefore, the following null hypotheses could not be rejected.

> There will not be a significant positive correlation between image acquisitions and concept formation.

Table 16. ANCOVA for Concept Acquisition

| Source | DF | SS | Mean Square | F Ratio | p |
|---|---|---|---|---|---|
| Pretest | 1 | .20 | .20 | .06 | 0.803 |
| Gender | 1 | 16.66 | 16.66 | 5.15* | 0.026 |
| Group | 2 | 25.23 | 12.62 | 3.90* | 0.025 |
| Gender*Group | 2 | .08 | .04 | .01 | 0.988 |
| Model | 6 | 43.79 | 7.30 | 2.26* | 0.047 |
| Error | 73 | 236.10 | 3.23 | | |

Note: *significant for p=.05

Because there was no significant correlation between the pretest and posttests, the covariate was dropped from the model and ANOVA's were run in image acquisition and concept acquisition. The results are reported in Table 17 and Table 18. The results indicate that males outperformed females in concept acquisition. Also, there was a significant difference among groups in concept acquisition. A Tukey HSD follow-up test revealed that the interactive-animated group scored higher than the static group on concept acquisition (p = .015) with a mean difference of 1.35 points.

Table 17. ANOVA for Image Acquisition

| Source | DF | SS | Mean Square | F Ratio | p |
|---|---|---|---|---|---|
| Gender | 1 | 19.09 | 19.09 | 3.82 | 0.055 |
| Group | 2 | 8.80 | 4.40 | 0.88 | 0.419 |
| Gender*Group | 2 | 0.27 | 0.13 | .03 | 0.974 |
| Model | 5 | 27.30 | 5.46 | 1.09* | 0.373 |
| Error | 74 | 370.39 | 5.01 | | |

Table 18. ANOVA for Concept Acquisition

| Source | DF | SS | Mean Square | F Ratio | p |
|---|---|---|---|---|---|
| Gender | 1 | 17.52 | 17.52 | 5.49* | 0.022 |
| Group | 2 | 25.36 | 12.68 | 3.97* | 0.023 |
| Gender*Group | 2 | 0.09 | 0.04 | 0.01 | 0.986 |
| Model | 5 | 43.59 | 8.72 | 2.73* | 0.026 |
| Error | 74 | 236.30 | 3.19 | | |

Note: *significant for p=.05

Because there was no significant interaction between gender and treatment, the interaction term was dropped from the model and ANOVA's were run on image acquisition and concept acquisition. The results are reported in Table 19 and Table 20. The results indicate that, in addition to the outcomes reported from the ANOVA with the interaction term included, males outperformed females in image acquisition.

Table 19. ANOVA for Image Acquisition (interaction term removed)

| Source | DF | SS | Mean Square | F Ratio | P |
|---|---|---|---|---|---|
| Gender | 1 | 19.64 | 19.64 | 4.03* | 0.048 |
| Group | 2 | 9.09 | 4.54 | 0.93 | 0.398 |
| Model | 3 | 27.03 | 9.01 | 1.85 | 0.146 |
| Error | 76 | 370.66 | 4.88 | | |

Note: *significant for p=.05

Table 20. ANOVA for Concept Acquisition (interaction term removed)

| Source | DF | SS | Mean Square | F Ratio | P |
|---|---|---|---|---|---|
| Gender | 1 | 17.54 | 17.54 | 5.64* | 0.020 |
| Group | 2 | 25.37 | 12.68 | 4.08* | 0.021 |
| Model | 3 | 43.50 | 14.50 | 4.66* | 0.005 |
| Error | 76 | 236.39 | 3.11 | | |

Note: *significant for p=.05

## Delimitations

1. Students that enroll in calculus are not representative of all mathematics students. Results of this study will only be generalizable only to calculus students. More specifically, since

women taking calculus are not representative of all female mathematics students, it is possible that there is no gender difference of spatial visualization ability among calculus students. A result in no significant interaction between gender and treatment in calculus classes cannot be generalized to lower-level mathematics classes.

2. The computer lesson developed for this study is short. The results may differ when students are exposed to computer animation and interactivity over a longer period of time.
3. The computer lesson developed for this study is limited in scope of concepts covered. The results may differ when the computer tutorials are part of a larger program that utilizes animated and interactive computer presentations of related mathematical concepts throughout the semester.
4. Some students were not attentive when working through the computer program. Also, some students may not have performed their best when taking the tests. Providing some external incentive (for example, a quiz grade) for good performance might have ameliorated this problem.
5. The posttest may not have accurately measured image acquisition. For example, students who responded incorrectly to the direction of rotation of the secant line (clockwise or counterclockwise) may not have known which direction was clockwise.

6. The posttest, a pencil and paper test, was more closely aligned with the instruction for the static group than the other groups.

# CHAPTER 5
# CONCLUSIONS AND DISCUSSION

## Summary

Static media, such as textbooks and chalkboards, have traditionally been used to describe dynamic images to the student. However, new technologies have emerged that allow students to view and interact with animated images. The purpose of this study was, first, to compare the effectiveness of static, animated, and interactive-animated presentations in imparting a mental image of the rotating secant line. Second, this study was designed to investigate the question of whether students who are successful at acquiring this mental image are more likely to achieve a deeper understanding of the tangent concept. Finally, this researcher considered the question of whether an interaction between gender and treatment exists.

This investigation was grounded in the Visual/Analytic theory (Zaskis et al., 1996), which hypothesizes that students successfully learn many mathematical concepts by alternating their attention between the visual and analytic aspects of the concept. With each alternation between the visual and analytic, the image takes on enhanced meaning and the distinction between the visual and analytic narrows.

The study was carried out in five first-semester calculus classes at Shippensburg University. All spring semester Calculus I classes participated in the study. Each of the 82 students was randomly assigned to one of three groups: Static, Animated, or Interactive-animated. Three different versions of computer software were designed by the researcher to present a lesson on the tangent as the limiting position of a rotating secant. In crafting each lesson, the researcher attempted to emphasize the connections between the graphical and analytical aspects of the tangent concept. In the static version of the lesson, the student was presented a graphic and text describing the rotation of a secant line that would take place if one intersection of the secant line with a graph moved closer to the other intersection of the secant with the graph. The animated version displayed a moving picture of the rotating secant line and highlighted the diminishing $\Delta x$ with a thickened blue line. The interactive-animated version presented a moving picture of the rotating secant line, which the student controlled by changing $\Delta x$ with the mouse.

All subjects took the Card Rotations test as a measure of spatial visualization ability. The posttest, divided into two parts, measured the constructs Image Acquisition and Concept Acquisition. A 3X2 ANCOVA was used to compare the groups. Statistical analysis revealed no significant differences among the groups in image acquisition.

Therefore, the following null hypotheses could not be rejected.

- The animated group will not outperform the static group in image acquisition.
- The interactive-animated group will not outperform the animated group in image acquisition.

There were no significant interactions between gender and treatment. Therefore the following null hypotheses could not be rejected:

- Females will not benefit more than males from animated presentations in image acquisition.
- Females will not benefit more than males from the interactive presentation in image acquisition.

Finally, because the correlation between image acquisition and concept acquisition was not statistically significant, the following null hypothesis could not be rejected:

- There will not be a significant positive correlation between image acquisitions and concept acquisition.

There was a significant difference among the groups in concept acquisition. Tukey's pairwise comparison indicated that the interactive-animated group outperformed the static group on this measure.

## Discussion

Results of the present study indicated no interaction between gender and treatment. I hypothesized an interaction would occur because prior studies have demonstrated that females have lower spatial visualization ability than males. Those with high spatial ability are likely to form a rich mental image of a rotating secant line from a static presentation. Those with low spatial ability should benefit from seeing an explicit motion picture of the image. However, in this study there was no significant difference between males and females in spatial ability or image acquisition. So, while an interaction is likely to occur between groups with differing spatial abilities, we did not see an interaction between gender and treatment.

This study revealed no treatment effect in image acquisition and no significant correlation between image acquisition and concept acquisition. Yet, the interactive-animated group outperformed the static group in concept acquisition. Scores on concept acquisition differed significantly between these two groups, with the interactive-animated group scoring over 31 percent higher than the static group. The animated group scored 14.5 percent higher than the static group on concept acquisition, but this was not a significant difference. Analysis of the data revealed a statistically significant treatment effect on concept acquisition (see Table 17). In retrospect, I should have hypothesized the above results in the beginning of this study; this hypothesis would be a logical conclusion based on hypotheses #1, #2, and #5 below.

1. The animated group will outperform the static group in image acquisition.
2. The interactive-animated group will outperform the animated group in image acquisition.

5. There will be a significant positive correlation between image acquisitions and concept formation.

The logical conclusions of these hypotheses are that:

- The animated group will outperform the static group in concept acquisition.
- The interactive-animated group will outperform the animated group in concept acquisition.

One might argue that the last two post hoc conclusions are not warranted in view of the facts that 1.) the correlation between image acquisition and concept acquisition was only moderately positive and not statistically significant and 2.) there were no group differences on the image acquisition posttest. However, I contend that there are different ways of seeing a rotating secant line. One explanation of the treatment effect on concept acquisition is that the students in the inactive-animated group were able to see the rotating secant image with deeper understanding, and were better able to abstract the relevant mathematical attributes of the image.

The V/A Theory supports my contention that there should be a correlation between image acquisition and concept acquisition. If one does not see the image of the rotating secant, he or she is deprived of an important visual image associated with the tangent concept. However, further analysis of the data from the present study suggests that most of the subjects from all groups did actually form an image of a rotating secant. When presented with the graph shown in Figure 31, 98 percent of all students were able to correctly answer that Δy got smaller as point Q approached point P. When presented with the graph in Figure 32, 88 percent of the subjects where able to correctly answer that the line PQ rotated in a clockwise direction as point Q approached point P.

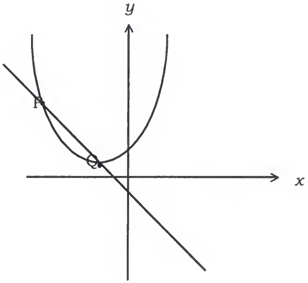


Figure 31. Ninety-eight percent of the students correctly answered that Δy got smaller as point Q approached point P.

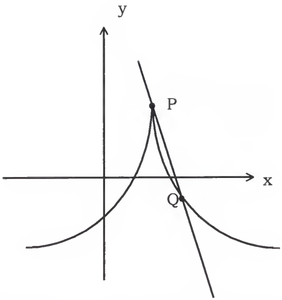


Figure 32. Eighty-eight percent of the students correctly answered that line PQ rotated in a clockwise direction as point Q approached point P.

The high proportion of correct answers to these two questions suggests that most of the subjects saw a rotating secant line and a diminishing $\Delta y$. However, the question remains, "Why did the interactive-animated group outperform the other two groups in concept acquisition?"

Some visual images are extremely helpful when learning mathematical concepts. However, people view images with differing levels of understanding. Just as an astronomer sees more in the night sky than a novice sees, a mathematician sees more in a graph than the untrained observer sees. In the case of the rotating secant line, some students may invest more meaning in the image than others may.

People view the same image with multifarious levels of understanding. When my wife and I go to an art museum and view a painting by Monet, we each see the painting in a different way. While outwardly our descriptions of the painting may be similar, she sees much more in the painting than I do. For example, if we view the painting *Water Lilies* both of our descriptions will include the flowers, water, and lily pads. But my wife will also notice the brush techniques, shading, type of paint used and many other aspects that I will overlook. Finally, she will recognize this painting as an extraordinary work of art; I will see the painting as a pretty picture. Our different view of *Water Lilies* is analogous to students' differing views of mathematical images. Outwardly, students' descriptions of the rotating secant line may be similar, but some students will invest more meaning in the image than others will.

Magidson (as cited in Moschkovich, Schoenfeld, & Arcavi, 1993) instructed algebra students to use graphing calculators to produce several graphs of $y = mx + b$ by increasing m and holding b constant. When asked, "What do you notice?" the most common response was that "The line gets more jagged." Orton (1980) showed calculus students a picture of a secant line that intersected a circle at points P and Q. When asked what happens to the secant as point Q approached point P, over 40 percent of the students were unable to say that the secant line tended toward a tangent. Typical responses were that the line became shorter or it disappeared. Both Magidson's students who saw a line becoming more jagged as its slope increased and Orton's students who saw a chord disappearing as one endpoint approached the other gave accurate descriptions of an image. But the algebra students who saw a line becoming more vertical as the coefficient in the linear equation was increased were more likely to understand the relationship between the coefficients in linear equations and their graphical representations. Similarly, the calculus students in Orton's study who saw a chord tending toward a tangent were more likely to have understood the teacher's previous discussions on the rotating secant line. The more enlightened students were able to notice the relevant aspects of the image. I hypothesize that the students in the interactive-animated group learned to see the image of the rotating secant differently than the students in the other groups. Perhaps the subjects in the interactive-animated group were better able to attend to the most pertinent aspects of the image. They were able to see not only a rotating secant line, but also see a functional relationship between $\Delta x$ and the position of the secant line. Students in the interactive-animated group were also able to see that as $\Delta x$ approached zero, the secant approached the tangent.

According to the VA Theory, visualization is a necessary, but not sufficient, component for understanding many mathematical ideas. Because the student must alternate between the visual and analytical aspects of a concept when gaining understanding, students who are stranded on the visual side will not progress toward a full understanding. The van Hiele model of development of geometric understanding is consistent with this. The first level (Level Zero) of geometric understanding, according to the van Hiele model, is visualization (Burger & Shaughnessy, 1986). It is not until Level Two that learners begin to understand the interrelationships of attributes within figures or that definitions make sense. Further, the van Hiele model asserts that learners must attain Level Zero, visualization, in order to reach the higher levels that include seeing the relationships between parts of a figure and understanding definitions. In the VA Theory, each alternation between visual and analytic produces a more refined and meaningful visualization. That is, after viewing a figure followed by introspection and analytic thought, the student then views the figure differently – with more understanding. The van Hiele theory suggests that when students reach Level One (Analysis) they see figures differently – and with more understanding. For example, they notice different parts of figures that were not discernible before. Both the V/A and the van Hiele models of mathematical learning support the notion of a hierarchy of levels of visualization. Different students view images with different levels of understanding.

Past research has demonstrated that many students have trouble incorporating visualizations into mathematical thinking.

Students often see the analytic and graphic representations of functions as separate and disconnected (Presmeg, 1986); students have difficulties understanding diagrams (Goldenberg, 1991, Kaput, 1989); and students are reluctant to use visual images when solving problems (Vinner, 1989). Mathematicians make frequent use of visual imagery while engaged in mathematical activity (Dreyfus, 1991, Hadamard, 1954, Davis & Anderson, 1979). But mathematicians, through training and experience, know what to look for in diagrams and are able to make generalizations from a particular diagram. So, simply making a powerful visual image available to the student may not be enough to elucidate a complex mathematical idea. The student also may need help in making sense of the diagram. I suggest that the interactive-animated version of the rotating secant line made the pertinent aspects of the diagram salient. The rhythm of the moving points and lines may have helped students who controlled the animations make the connections between the relevant aspects of the image. Change $\Delta x$ quickly and the secant line rotates quickly. Increase $\Delta x$ and the secant rotates one way; decrease $\Delta x$ and the secant line rotates the other. These correspondences, which add the dimensions of speed and direction, make the functional relationships between important parts of the animation evident. In other words, while it appears that there is no difference conveying the visual image of the rotating secant line with the static or interactive-animated medium, it is possible that the interactive-animated presentation imparted the image with enhanced meaning.

My contention, then, is that the interactive-animated version imparted a visual image with more meaning by making the pertinent aspects of the figure more salient. But I think there is more to the interactive version of the image that helps students understand. When the student interacts with and controls the moving image, a kind of gestalt results; he or she sees the moving parts of the image as a connected whole. The student who sees the image as a whole is able to see the dependence among all of the moving parts. Cottril et al. (1996) suggested that an understanding of $\lim_{x \to a} f(x) = L$ requires a coordination of two dynamic concepts: (1) $x$ approaching a, and (2) $f(x)$ approaching $L$. Most students are unable to achieve this coordination of concepts, and this failure is the primary impediment to understanding limits. Even though Cottril's subjects could see $x$ approaching $a$ AND $f(x)$ approaching $L$, many students missed the connection between the two dynamic processes.

While a student might have had an image of $\Delta x$ approaching zero and the secant approaching the tangent, it is the coordination of the two events that is necessary for the understanding of the limit of the difference quotient. If Cortril et al. (1996) are correct about the importance of coordination of dynamic concepts in understanding limits, it is plausible that an even more difficult coordination of four dynamic events are necessary for understanding the tangent concept:

1. Point Q approaching point P
2. $\Delta x$ approaching zero
3. $\Delta y$ approaching zero
4. The secant line approaching the tangent

So, while the results of the image acquisition test demonstrated no significant differences among the groups in acquiring images 1 through 4 above, the test provided no indication of which students were able to effectively coordinate these four dynamic ideas. If one extends the theory of Cottril et al. (1996) to the tangent concept, it is plausible that those who best coordinated these concepts scored highest on the concept acquisition measure. It is possible, therefore, that the more concrete, interactive-animated representation of the rotating secant image is helpful to students in coordinating the events necessary for understanding the tangent concept. The rhythm of the moving lines may have helped students in the interactive-animated group make the connections between the relevant aspects of the image. Change $\Delta x$ quickly and the secant line rotates quickly. Increase $\Delta x$ and the secant rotates one way; decrease $\Delta x$ and the secant line rotates the other. These correspondences make the functional relationship between important aspects of the animation evident and may have helped the students coordinate the four dynamic events.

The interactive-animated version of the software provided a richer learning experience. The use of the mouse in controlling the animation involves both the students' musculature and tactile sense. Adding these kinesthetic and sensory elements to the experience may have allowed for a more meaningful encounter. The students who interact with an image on the computer will come away with a deeper understanding of the tangent concept than the students who only view an image. I hypothesize that the reason for this is that students who interact with the image are more likely to attend to the pertinent aspects of the image, coordinate the functional relationships of its moving parts, and form a gestalt in which the elements in motion come together as a connected whole rather than separate, vaguely related movements. The multi-sensory experience of interacting with the image will result in a more meaningful experience. Miller (1990) reflected on the experience of watching a drive-in movie from afar. In a prescient observation, he pointed out that without the benefit of sound, one sees a choppy series of images and notices that the video rapidly jumps from one scene to the next. The movie with sound is viewed quite differently as "one becomes engrossed in the story and the continuity of meaning overwhelms the discontinuity of presentation" (p. 190). A similar phenomenon occurs in the interactive version of the rotating secant image. The student becomes engrossed in the image, and the image takes on enhanced meaning.

## Suggestions for further study

The capabilities of the computer are growing at a much faster rate our understanding of how to use computers in mathematics education. After all, there are many more people and much more capital invested in improving computers than in research on the use of computers in mathematics education. Intel, alone, spent $2.3 billion for research and development in 1998 (1998 Intel Annual Report). Because computers offer a vast potential for enhancing mathematics learning, I suggest a greater emphasis on research in the productive use of computers in mathematics education. The present study suggests several areas of related research:

1. Study the benefits of animated and interactive computer imagery in areas other than calculus learning and for populations other than calculus students. For example, I suggest a study comparing static, animated, and interactive-animated instructional treatments in teaching transformations of trigonometric functions.
2. Study the interaction between spatial ability and treatments involving the animation and interactive capabilities of the computer.

3. Investigate the kinesthetic elements of mathematical thinking while working in a computer assisted learning environment. There is a scarcity of research on kinesthetic aspects of learning mathematics. Although mathematicians, educators, and psychologist make reference to this manner of learning, few studies have been conducted that attempt to isolate this variable as an element of learning mathematical ideas or problem solving. The use of the mouse involves the musculature while interacting with computer images.
4. Study the benefits of adding narration to animated and interactive-animated computer presentations of mathematical ideas.

Expand the present study in the following ways:

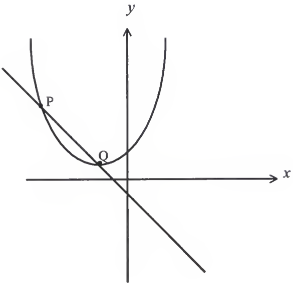
1. I posit that the treatment effect in the present study on concept acquisition occurred, in part, because more students in the interactive-animated group than in the static group were attending to the pertinent attributes of the image. I suggest a qualitative study with the goal of identifying different concept images among students exposed to static, animated and interactive animated presentations of the tangent concept.

   Another way to carry out this study is in a test-treatment-retest format. Students receive the static treatment, are tested and interviewed, then receive the interactive-animated treatment, followed by a retest and interview. As a control, some students can receive the static treatment twice.

2. The present study involved university calculus students. I propose a similar study be carried out to compare acquisition of the tangent concept among static, animated, and interactive-animated groups in community college students.

# APPENDIX
# POSTTEST

Posttest Name ______________________________



Refer to the graph above and identify the following as true or false

As Q approaches P:

1. $\Delta$x gets smaller. TRUE or FALSE

2. $\Delta$y gets smaller. TRUE or FALSE

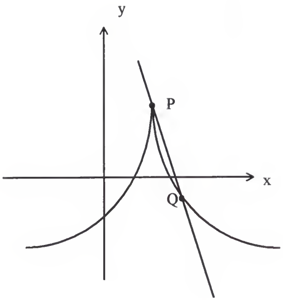
3. The slope of the secant line increases. TRUE or FALSE

4. The line PQ rotates in a clockwise fashion. TRUE or FALSE

5. The slope of the line PQ increases. TRUE or FALSE

6. $\frac{\Delta y}{\Delta x}$ approaches infinity. TRUE or FALSE

7. $\frac{\Delta y}{\Delta x}$ approaches zero. TRUE or FALSE



Refer to the graph above and identify the following as true or false

As Q approaches P:

8. Δx gets smaller. TRUE or FALSE

9. Δy gets smaller. TRUE or FALSE

10. The slope of the secant line increases. TRUE or FALSE

11. The line PQ rotates in a clockwise fashion. TRUE or FALSE

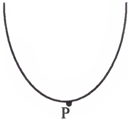
12. The slope of the line PQ increases. TRUE or FALSE

13. $\frac{\Delta y}{\Delta x}$ approaches infinity. TRUE or FALSE

14. $\frac{\Delta y}{\Delta x}$ approaches zero. TRUE or FALSE

Below are two curves. On each one of them a point P is denoted. Below each one of them are three statements. Circle the statement which seems true to you and follow any instruction in parentheses.

15.

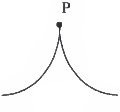


A. Through P it is possible to draw exactly one tangent to the curve (draw it).

B. Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinitely many. Draw all of them if their number is finite and some of them if it is infinite).

C. It is impossible to draw through P a tangent to the curve.

16.



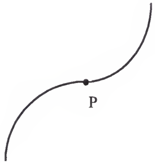
A. Through P it is possible to draw exactly one tangent to the curve (draw it).

B. Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinitely many. Draw all of them if their number is finite and some of them if it is infinite).

C. It is impossible to draw through P a tangent to the curve.

Below are two curves. On each one of them a point P is denoted. Below each one of them are three statements. Circle the statement which seems true to you and follow any instruction in parentheses.

17.

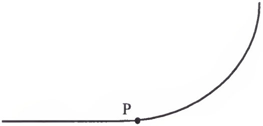


A. Through P it is possible to draw exactly one tangent to the curve (draw it).

B. Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinitely many. Draw all of them if their number is finite and some of them if it is infinite).

C. It is impossible to draw through P a tangent to the curve.

18.

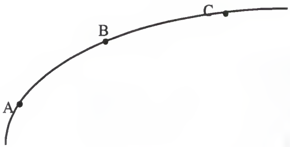


A. Through P it is possible to draw exactly one tangent to the curve (draw it).

B. Through P it is possible to draw more than one tangent (specify how many, one, two, three, infinitely many. Draw all of them if their number is finite and some of them if it is infinite).

C. It is impossible to draw through P a tangent to the curve.

19. Which point on the graph below is its slope the greatest?



A. Point A B. Point B C. Point C

20. Which of the following is the correct slope of the tangent to $y = g(x)$ at the point (a,b)?

A. $\lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(b)}{\Delta x}$

B. $\lim_{\Delta x \to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x}$

C. $\lim_{\Delta x \to 0} \frac{g(b - \Delta x) - g(b)}{\Delta x}$

D. $\lim_{\Delta x \to 0} \frac{g(b + \Delta x) - g(b)}{\Delta x}$

E. None of the above are the correct slope.

21. Which of the following is the correct slope of the tangent to $a = v(t)$ at the point (c,d)?

A. $\lim_{\Delta a \to 0} \frac{v(a + \Delta a) - v(a)}{\Delta a}$

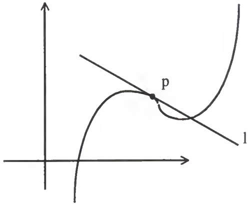
B. $\lim_{\Delta d \to 0} \frac{v(d + \Delta d) - v(d)}{\Delta d}$

C. $\lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x}$

D. $\lim_{\Delta t \to 0} \frac{v(c + \Delta t) - v(t)}{\Delta t}$

E. None of the above are the correct slope.

22. Line l is tangent to the graph of $y = f(x)$ at point p. TRUE or FALSE



23. What is the slope of the tangent to the graph of $y = x^2$ at the point $(p,q)$?

A. $\lim_{\Delta x \to 0} \frac{(p+\Delta x)^2 - p^2}{\Delta x}$

B. $\lim_{\Delta x \to 0} \frac{(p+\Delta x)^2 - p}{\Delta x}$

C. $\lim_{\Delta x \to 0} \frac{(q+\Delta q)^2 - q^2}{\Delta x}$

D. $\lim_{\Delta p \to 0} \frac{(p+\Delta p)^2 - p^2}{\Delta p}$

E. None of the above are the correct slope.

24. Which one of the following statements is correct?

A. The slope of a tangent to a straight line is equal to the slope of the straight line.

B. The tangent to a straight line does not exist.

C. There is more than one tangent to a straight line at any given point on the line.

D. The slope of a tangent to a straight line is always zero.

E. None of the statements are correct.

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